7
MOTION IN A SCHWARZSCHILD METRIC

7.1 General Motion in a Schwarzschild Metric

In this section, we examine the effect of the form of the Schwarzschild metric on the inverse r^2 gravitational interaction between two masses to an accuracy of v^2/c^2. Our subsequent calculations regarding the advance of the perihelion of Mercury are carried out in flat space. Since we live in curved space, all calculations should , in principle, be carried out *ab initio* in that space. However, in flat space, corrections are easier to make and yield an accuracy of v^2/c^2 which is sufficient to account for the observed perihelion advance, a goal of the present work. The same calculations in curved space would simply add higher order terms. We will now show, however, that curvature terms in the Schwarzschild metric itself produces terms that retard the period by a correction of v^2/c^2. This correction must be subtracted from the flat space corrections to obtain the net perihelion advance.

The following relations will be used:

$$-\frac{GMm}{r^2} = -\frac{mv^2}{r} \qquad \frac{GM}{r^2} = \frac{v^2}{r} = a = \ddot{r} \qquad \frac{GM}{c^2 r} = \frac{v^2}{c^2}$$

$$\frac{2\alpha}{r} = \frac{2GM}{rc^2} = \frac{2v^2}{c^2} \qquad \alpha = \frac{GM}{c^2} \qquad \frac{\alpha}{r} = \frac{v^2}{c^2}$$

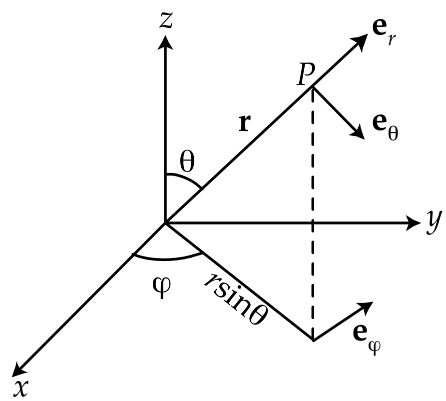

Fig. 7.1

In general, a differential space-time interval may be written

$$d\mathbf{s} = \mathbf{e}_0 c h_t dt + \mathbf{e}_r h_r dr + \mathbf{e}_\theta h_\theta d\theta + \mathbf{e}_\varphi h_\varphi d\varphi \tag{7.1}$$

$\mathbf{e}_0,\ \mathbf{e}_r,\ \mathbf{e}_\theta,\ \mathbf{e}_\varphi$ are unit vectors.

Letting

$$h_t = (1 - 2\alpha/r)^{1/2} \qquad h_r = (1 - 2\alpha/r)^{-1/2} \qquad h_\theta = r, \qquad h_\varphi = r\sin\theta$$

gives the differential Schwarzschild space-time interval

$$d\mathbf{s} = \mathbf{e}_0 (1 - 2\alpha/r)^{1/2} c dt + \mathbf{e}_r (1 - 2\alpha/r)^{-1/2} dr + \mathbf{e}_\theta r d\theta + \mathbf{e}_\varphi r \sin\theta d\varphi \tag{7.2}$$

Divide ds by dt to obtain the space-time velocity vector

$$\mathbf{V} = \frac{d\mathbf{s}}{dt} = \mathbf{e}_0 (1 - 2\alpha/r)^{1/2} c + \mathbf{e}_r (1 - 2\alpha/r)^{-1/2} \frac{dr}{dt} + \mathbf{e}_\theta r \frac{d\theta}{dt} + \mathbf{e}_\varphi r \sin\theta \frac{d\varphi}{dt} \tag{7.3}$$

Differentiate $\mathbf{V}$ in (7.3) to obtain the space-time acceleration:

$$\begin{aligned}
\frac{d\mathbf{V}}{dt} &= \mathbf{e}_0 \frac{d}{dt}(c h_t) + \frac{d}{dt}(\mathbf{e}_r h_r \dot{r}) + \frac{d}{dt}\left(\mathbf{e}_\theta h_\theta \dot{\theta}\right) + \frac{d}{dt}(\mathbf{e}_\varphi h_\varphi \dot{\varphi}) \\
\frac{d\mathbf{V}}{dt} &= \mathbf{e}_0 \frac{d}{dt}(1 - 2\alpha/r)^{1/2} + \frac{d}{dt}\left[\mathbf{e}_r (1 - 2\alpha/r)^{-1/2} \dot{r}\right] \\
&\quad + \frac{d}{dt}\left(\mathbf{e}_\theta r \dot{\theta}\right) + \frac{d}{dt}(\mathbf{e}_\varphi r \sin\theta\ \dot{\varphi})
\end{aligned} \tag{7.4}$$

$$\begin{aligned}
&\frac{d\mathbf{V}}{dt} = \mathbf{e}_0 \frac{d}{dt} c (1 - 2\alpha/r)^{1/2} \\
&+ \mathbf{e}_r \frac{d}{dt}\left[(1 - 2\alpha/r)^{-1/2}\right] \dot{r} + \mathbf{e}_r (1 - 2\alpha/r)^{-1/2} \ddot{r} + \frac{d\mathbf{e}_r}{dt} r (1 - 2\alpha/r)^{-1/2} \dot{r} \\
&+ r\dot{\theta} \frac{d\mathbf{e}_\theta}{dt} + \mathbf{e}_\theta \frac{dr}{dt}\dot{\theta} + \mathbf{e}_\theta r \ddot{\theta} + (r\sin\theta)\dot{\varphi} \frac{d\mathbf{e}_\varphi}{dt} + \mathbf{e}_\varphi \frac{d(r\sin\theta)}{dt}\dot{\varphi} + \mathbf{e}_\varphi (r\sin\theta)\ddot{\varphi}
\end{aligned} \tag{7.5}$$

Derivatives of the unit vectors in Eq. (7.5) are:

$$
\begin{aligned}
\frac{\partial \mathbf{e}_r}{\partial \theta} &= \mathbf{e}_\theta \quad \frac{\partial \mathbf{e}_r}{\partial \varphi} = \mathbf{e}_\varphi \sin\theta & \frac{\partial \mathbf{e}_r}{\partial r} &= 0 \\
\frac{\partial \mathbf{e}_\theta}{\partial \theta} &= -\mathbf{e}_r \quad \frac{\partial \mathbf{e}_\theta}{\partial \varphi} = \mathbf{e}_\varphi \cos\theta & \frac{\partial \mathbf{e}_\theta}{\partial r} &= 0 \\
\frac{\partial \mathbf{e}_\varphi}{\partial \theta} &= 0 \quad \frac{\partial \mathbf{e}_\varphi}{\partial \varphi} = -\mathbf{e}_\theta \cos\theta - \mathbf{e}_r \sin\theta & \frac{\partial \mathbf{e}_\varphi}{\partial r} &= 0 \\
\frac{d\mathbf{e}_r}{dt} &= \frac{\partial \mathbf{e}_r}{\partial \theta}\dot{\theta} + \frac{\partial \mathbf{e}_r}{\partial \varphi}\dot{\varphi} + \frac{\partial \mathbf{e}_r}{\partial r}\dot{r} = \mathbf{e}_\theta\dot{\theta} + \mathbf{e}_\varphi\dot{\varphi}\sin\theta, & \frac{\partial \mathbf{e}_r}{\partial r}\dot{r} &= 0 \\
\frac{d\mathbf{e}_\theta}{dt} &= \frac{\partial \mathbf{e}_\theta}{\partial \theta}\dot{\theta} + \frac{\partial \mathbf{e}_\theta}{\partial \varphi}\dot{\varphi} + \frac{\partial \mathbf{e}_\theta}{\partial r}\dot{r} = -\mathbf{e}_r\dot{\theta} + \mathbf{e}_\varphi\dot{\varphi}\cos\theta, & \frac{\partial \mathbf{e}_\theta}{\partial r}\dot{r} &= 0 \\
\frac{d\mathbf{e}_\varphi}{dt} &= \frac{\partial \mathbf{e}_\varphi}{\partial \theta}\dot{\theta} + \frac{\partial \mathbf{e}_\varphi}{\partial \varphi}\dot{\varphi} + \frac{\partial \mathbf{e}_\varphi}{\partial r}\dot{r} = -\mathbf{e}_\theta\dot{\varphi}\cos\theta - \mathbf{e}_r\dot{\varphi}\sin\theta, & \frac{\partial \mathbf{e}_\varphi}{\partial r}\dot{r} &= 0
\end{aligned}
\tag{7.6}
$$

Inserting the derivatives of the unit vectors in (7.6) into Eq. (7.5), we obtain

$$
\begin{aligned}
\frac{d\mathbf{V}}{dt} &= \mathbf{e}_0 c \frac{d}{dt}(1-2\alpha/r)^{1/2} \\
&+ (1-2\alpha/r)^{-1/2}\,\dot{r}\left(\mathbf{e}_\theta\dot{\theta} + \mathbf{e}_\varphi\dot{\varphi}\sin\theta\right) \\
&+ \mathbf{e}_r\left[\frac{d}{dt}(1-2\alpha/r)^{-1/2}\right]\dot{r} + \mathbf{e}_r(1-2\alpha/r)^{-1/2}\,\ddot{r} \\
&+ r\dot{\theta}\left(-\mathbf{e}_r\dot{\theta} + \mathbf{e}_\varphi\dot{\varphi}\cos\theta\right) + \mathbf{e}_\theta\dot{r}\dot{\theta} + \mathbf{e}_\theta r\ddot{\theta} \\
&+ r\sin\theta\dot{\varphi}\left(-\mathbf{e}_\theta\dot{\varphi}\cos\theta - \mathbf{e}_r\dot{\varphi}\sin\theta\right) \\
&+ \mathbf{e}_\varphi\left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)\dot{\varphi} + \mathbf{e}_\varphi r\sin\theta\ddot{\varphi}
\end{aligned}
\tag{7.7}
$$

Collecting coefficients of the unit vectors we obtain

$$
\begin{aligned}
\frac{d\mathbf{V}}{dt} &= \mathbf{e}_0 c \frac{d}{dt}(1-2\alpha/r)^{1/2} \\
&+ \mathbf{e}_r\left[\frac{d}{dt}(1-2\alpha/r)^{-1/2}\,\dot{r} + (1-2\alpha/r)^{-1/2}\left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta\right)\right] \\
&+ \mathbf{e}_\theta\left[(1-2\alpha/r)^{-1/2}\,\dot{r}\dot{\theta} + \underline{\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\varphi}^2 + r\ddot{\theta}}\right] \\
&+ \mathbf{e}_\varphi\left[(1-2\alpha/r)^{-1/2}\,\dot{r}\dot{\varphi}\sin\theta + \underline{r\dot{\theta}\dot{\varphi}\cos\theta + \dot{r}\dot{\varphi}\sin\theta + r\dot{\theta}\dot{\varphi}\cos\theta + r\sin\theta\ddot{\varphi}}\right]
\end{aligned}
\tag{7.8}
$$

For $2\alpha/r = 0$, Eq. (7.8) reduces to

$$\frac{d\mathbf{V}}{dt} = \mathbf{e}_0\,(0) + \mathbf{e}_r\left[\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta\right] + \mathbf{e}_\theta\left[2\dot{r}\dot{\theta} - \dot{\varphi}^2 r\sin\theta\cos\theta + r\ddot{\theta}\right] + \mathbf{e}_\varphi\left[2\dot{r}\dot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\cos\theta + \ddot{\varphi}r\sin\theta\right] \tag{7.9}$$

Eq. (7.9) is the standard flat space expression for $d\mathbf{V}/dt$.

7.2 Planar Planetary Motion in a Schwarzschild Metric

For motion in a plane, let $\dot{\theta} = 0$, $\theta = \pi/2$, $\sin\theta = 1$, $\cos\theta = 0$. Then, in Eq. (7.48),

$$\frac{d\mathbf{V}}{dt} = \mathbf{e}_0\frac{d}{dt}c\,(1-2\alpha/r)^{1/2} + \mathbf{e}_r\left[\frac{d}{dt}\,(1-2\alpha/r)^{-1/2}\,\dot{r} + (1-2\alpha/r)^{-1/2}\left(\ddot{r} - r\dot{\varphi}^2\right)\right] + \mathbf{e}_\varphi\left[(1-2\alpha/r)^{-1/2}\,\dot{r}\dot{\varphi} + \dot{r}\dot{\varphi} + r\ddot{\varphi}\right] \tag{7.10}$$

Thus for the Schwarzschild metric

$$\frac{d\mathbf{V}}{dt} = \mathbf{e}_0\frac{c}{(1-2\alpha/r)^{1/2}}\frac{\alpha\dot{r}}{r^2} + \mathbf{e}_r\left(\frac{-\alpha\dot{r}^2}{(1-2\alpha/r)^{3/2}}\frac{1}{r^2} + \frac{\ddot{r} - r\dot{\varphi}^2}{(1-2\alpha/r)^{1/2}}\right) + \mathbf{e}_\varphi\left[\dot{r}\dot{\varphi} + \frac{\dot{r}\dot{\varphi}}{(1-2\alpha/r)^{1/2}} + r\ddot{\varphi}\right] \tag{7.11}$$

Removing the term in $\mathbf{e}_0$, the orbital value for $d\mathbf{V}/dt$ in a Schwarzschild metric becomes:

$$\frac{d\mathbf{V}}{dt} = \left[\mathbf{e}_r\left(\frac{\ddot{r}}{(1-2\alpha/r)^{1/2}} - \frac{\dot{r}^2\alpha}{(1-2\alpha/r)^{3/2}\,r^2} - \frac{r\dot{\varphi}^2}{(1-2\alpha/r)^{1/2}}\right) + \mathbf{e}_\varphi\left(\dot{r}\dot{\varphi} + \frac{\dot{r}\dot{\varphi}}{(1-2\alpha/r)^{1/2}} + r\ddot{\varphi}\right)\right] \tag{7.12}$$

Note that:

$$\mathbf{e}_0 mc\left[(1+\alpha/r)\left(\alpha\dot{r}/r^2\right)\right] \cong 0$$

When $\alpha = 0$, Eq. (7.12) gives the planar acceleration for flat space:

$$\frac{d\mathbf{V}}{dt} = \mathbf{a} = \mathbf{e}_0\,(0) + \mathbf{e}_r\left(\ddot{r} - r\dot{\varphi}^2\right) + \mathbf{e}_\varphi\left(2\dot{r}\dot{\varphi} + r\ddot{\varphi}\right) \tag{7.13}$$

When the force producing the acceleration is central, the coefficient of $\mathbf{e}_\varphi$ is zero, thus for planar motion $2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0$ which may be written

$$2\dot{r}\dot{\varphi} + r\ddot{\varphi} = \frac{1}{r}\frac{d\left(r^2\dot{\varphi}\right)}{dt} = 0 \tag{7.14}$$

Thus for a mass m

$$\frac{\left(mr^2\dot{\varphi}\right)}{r} = \frac{L_0}{r} = \text{const}$$

where L_0 is the angular momentum of a mass m at radius r. This is an initial condition.

From $L_0 = mr^2\dot{\varphi}$, $\dot{\varphi} = L_0/mr^2$. Then for planar motion,

$$m\mathbf{a} = \mathbf{e}_r m\left(\ddot{r} - r\dot{\varphi}^2\right) = \mathbf{e}_r m\left(\ddot{r} - r\left(\frac{L_0}{mr^2}\right)^2\right) = \mathbf{e}_r m\left(\ddot{r} - \frac{L_0^2}{mr^4}\right)$$

For a central force in the Schwarzschild metric, Eq. (7.12), put the coefficient of $\mathbf{e}_\varphi = 0$. Then, since $1/\left(1 - 2\alpha/r\right)^{1/2} \simeq 1 + \alpha/r$

$$\dot{r}\dot{\varphi} + \dot{r}\dot{\varphi}\left(1 + \frac{\alpha}{r}\right) + r\ddot{\varphi} = 0 \tag{7.15}$$

Eq. (7.15) approximately equals zero, but we equate it exactly equal to zero to obtain after multiplying it by r

$$r\left(2\dot{r}\dot{\varphi} + r\ddot{\varphi}\right) + r\dot{r}\dot{\varphi}\frac{\alpha}{r} = 0$$

Since

$$\begin{aligned} r\left(2\dot{r}\dot{\varphi} + r\ddot{\varphi}\right) &= \frac{d}{dt}\left(r^2\dot{\varphi}\right) \\ \frac{d}{dt}\left(r^2\dot{\varphi}\right) + r\dot{r}\dot{\varphi}\alpha &= 0 \end{aligned} \tag{7.16}$$

With the coefficient of $\mathbf{e}_\varphi$ zero Eq. (7.12) becomes

$$\frac{d\mathbf{V}}{dt} = \mathbf{e}_r\left[-\frac{\alpha\dot{r}^2}{\left(1 - 2\alpha/r\right)^{3/2}}\frac{1}{r^2} + \frac{\ddot{r}}{\left(1 - 2\alpha/r\right)^{1/2}} - \frac{r\dot{\varphi}^2}{\left(1 - 2\alpha/r\right)^{1/2}}\right] \tag{7.17}$$

The minus sign is inserted for the results of $d\mathbf{V}/dt$ since the acceleration produced is centripetal. This is the exact equation that describes planet motion in a Schwarzschild metric. For planar motion under a gravitational force, Eq. (7.17) may now be written

$$\begin{aligned}\frac{GMm\mathbf{e}_r}{r^2} &= \frac{d\mathbf{V}}{dt} \\ &\cong \mathbf{e}_r m\left[\ddot{r}\left(1+\frac{\alpha}{r}\right) - r\dot{\varphi}^2\left(1+\frac{\alpha}{r}\right) - \alpha\dot{r}^2\left(1+\frac{3\alpha}{r}\right)\frac{1}{r^2}\right] \quad (7.18)\end{aligned}$$

$$r\dot{\varphi}^2 = \frac{r^2\dot{\varphi}^2}{r} = \frac{v^2}{r} \qquad \dot{\varphi} = \frac{v}{r}$$

$$\frac{\alpha}{r} = \frac{GM}{r^2} = \frac{v^2}{c^2}$$

Therefore

$$\mathbf{F} = -\frac{\mathbf{e}_r mGM}{r^2} = -\mathbf{e}_r m\left[\ddot{r}\left(1+\frac{\alpha}{r}\right) - r\left(\frac{L_0}{mr}\right)^2 - \alpha\dot{r}^2\left(1+\frac{3\alpha}{r}\right)\frac{1}{r^2}\right] \quad (7.19)$$

The last two terms in Eq. (7.19) are also small. $\mathbf{e}_r mGM/r^2$ is centripetal, the same as $\ddot{r}$ is for central force motion.

Thus for an accuracy of v^2/c^2, Eq. (7.19) becomes

$$\frac{GM}{r^2} = \ddot{r}\left(1+\frac{\alpha}{r}\right) = \ddot{r}\left(1+\frac{v^2}{c^2}\right)$$

or

$$\frac{GM}{r^2\left(1+v^2/c^2\right)} \simeq \frac{MG\left(1-v^2/c^2\right)}{r^2} = \ddot{r} \quad (7.20)$$

Thus in the radial terms, the gravitational G in a Schwarzschild metric, is less than flat space G by the factor $(1-v^2/c^2)$ and the perihelion of Mercury is retarded rather than advanced by the fraction $(1-v^2/c^2)$.

We will now show how the angular momentum L in the Schwarzschild metric is related to the angular momentum L_0 in flat space. It will be shown that angular momentum in the Schwarzschild metric is less than that in flat space by a factor $(1-v^2/c^2)$. This correction is consistent with the radial retardation. It is not independent of it. The effect of the two corrections retards the advance by the factor $(1-v^2/c^2)$. The effect is a correction of $-\frac{v^2}{c^2}G$ to G. Thus the Schwarzschild metric causes a retrogression of v^2/c^2 per century or 7 seconds of arc per century.

In the following, we show that the Schwarzschild angular momentum L is less than the flat space angular momentum by the factor $(1-v^2/c^2)$, that is, $L = L_0\left(1-v^2/c^2\right)$, so it is consistent with the same radial factor.

Starting with

$$m\left(2r\dot{r}\dot{\varphi}+r^2\ddot{\varphi}+\dot{r}\dot{\varphi}\alpha\right)=0$$

$$\begin{aligned} m\left[\frac{d}{dt}\left(r^2\dot{\varphi}\right)+\dot{r}\dot{\varphi}\alpha\right] &= 0 \qquad\qquad \dot{r}\dot{\varphi}\alpha=\dot{r}\frac{L_0}{r^2}\alpha \\ \frac{d}{dt}L_0+\frac{\dot{r}}{r^2}\left(L_0\alpha\right) &= 0 \qquad\qquad \frac{\frac{dL_0}{dt}}{L_0}+\frac{\dot{r}}{r^2}\alpha=0 \\ \frac{dL_0}{L_0}+\frac{\dot{r}}{r^2}dt\alpha &= 0 \\ \frac{dL_0}{L_0}-\frac{\alpha d}{dt}\left(\frac{1}{r}\right)dt &= 0 \end{aligned}$$

$$d\log L_0-\alpha d\left(\frac{1}{r}\right)=0$$

$$\log L_0-\frac{\alpha}{r}=\text{const.}=\log L$$

$$\begin{aligned} \log\frac{L_0}{L} &= \frac{\alpha}{r} \qquad\qquad \text{Let}\quad L=L_0-x \\ \log\frac{L_0}{L_0-x} &= \frac{\alpha}{r} \qquad\qquad y=x/L_0 \\ \log\frac{1}{1-x/L_0} &= \frac{\alpha}{r} \\ \log\left(1+x/L_0\right) &= \frac{\alpha}{r} \\ \log\left(1+y\right) &= \frac{\alpha}{r} \\ \log\left(1+y\right) &= y-\frac{y^2}{2}\cong y \end{aligned}$$

$$\begin{aligned} \frac{x}{L_0} &= \frac{\alpha}{r} \qquad\qquad \frac{L_0-L}{L_0}=\frac{\alpha}{r} \\ L_0-L &= L_0\frac{\alpha}{r} \end{aligned} \tag{7.21}$$

Thus

$$L = L_0\left(1-\frac{\alpha}{r}\right)=L_0\left(1-\frac{v^2}{c^2}\right)$$

$$L = mr^2\dot{\varphi}\left(1-\frac{v^2}{r^2}\right) \tag{7.22}$$

and the angular momentum in the Schwarzschild metric is smaller than its value in the flat space metric. The period is

$$\begin{aligned} T &= \frac{2\pi r}{v} \qquad v = \text{ new velocity}, \quad v_0 = \text{ old velocity} \\ v &= \frac{L}{r} = r\dot{\varphi}\,(1-\alpha/r) = v_0\,(1-\alpha/r) \\ T &= \frac{2\pi r}{v_0\,(1-\alpha/r)} \cong \frac{2\pi r}{v_0}\left(1+\frac{a}{r}\right) = T_0\left(1+v^2/c^2\right) \end{aligned} \tag{7.23}$$

Thus the new period is v^2/c^2 longer than the flat space value.

7.3 Conclusion

The above results are equivalent to forming a G value of

$$G \to G\left(1-\frac{v^2}{c^2}\right) \tag{7.24}$$

This correction term is consistent with the radial correction term, Eq. (7.20), not independent of it. This correction term is listed as item 3 in the Summary provided in Section 2.4.