22

APPENDIX: ELECTROMAGNETISM. SOME BASICS

22.1 Review of a Few Introductory Topics in Electromagnetism.

In this section, we review a few common formulas in physics and give an alternate derivation of the angular momentum of two interacting particles.

1. (a) Flux flow argument leading to Gauss's law.

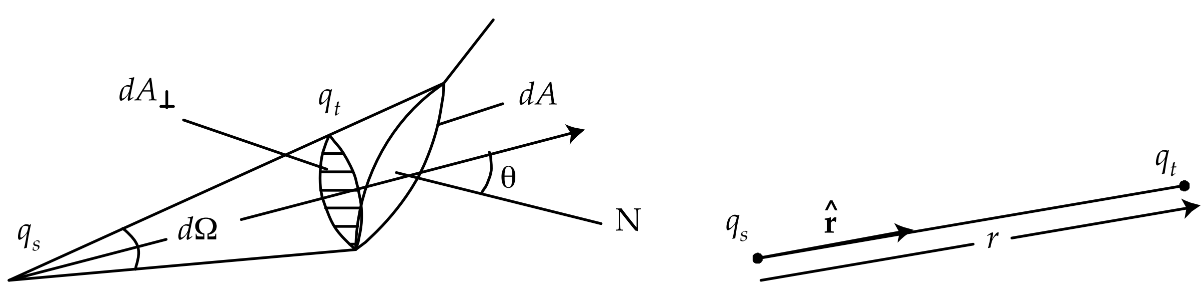

Fig. 22.1

Flux df from q_s is proportional to q_s and element of solid angle $d\Omega$. dA is element of area of arbitrary surface at distance r.

$$df = kq_s d\Omega \qquad d\Omega = \frac{dA_\perp}{r^2} \qquad k = \frac{1}{4\pi\varepsilon_0}$$

$$df = \frac{q_s}{4\pi\varepsilon_0}\frac{dA_\perp}{r^2} \tag{22.1}$$

$$\frac{df}{dA_\perp} = \frac{q_s\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2} = \mathbf{E} = \text{field strength} \tag{22.2}$$

Force of charge q_s on q_t is

$$\mathbf{F}_{ts} = \frac{df}{dA}q_t \to \frac{q_s q_t\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2} = \text{Coulomb's law} \tag{22.3}$$

$$k = \frac{1}{4\pi\varepsilon_0} = 9\times 10^9\frac{\text{Nm}^2}{\text{C}^2}$$

1. (b) Force of the field $\mathbf{E}$ on charge q_t

$$\mathbf{F} = \mathbf{E}q_t \qquad \mathbf{E} = \frac{\text{Newtons}}{\text{Coulomb}}$$

Work required to move charge q_t a distance $\mathbf{d}$

$$\text{Work} = \mathbf{F}\cdot\mathbf{d} = q_t\mathbf{E}\cdot\mathbf{d} = q_t E d\cos\theta$$

When $$E = 1\frac{\text{N}}{\text{Coul}} \qquad q_t = 1 \text{ Coul} \quad \text{and} \quad d = 1 \text{ meter}$$

$$\text{W} = 1 \text{ volt potential difference}$$

1. (c)

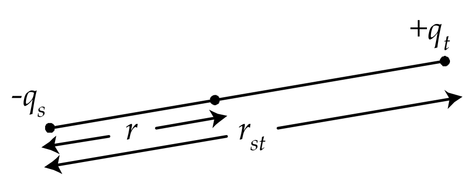

Fig 22.2

Work required to move q_t from r to r_{st}

$$\text{W} = \int F dr = \frac{1}{4\pi\varepsilon_0}\int_r^{r_{st}}\frac{q_s q_t}{r^2}dr = \left[-\frac{q_s q_t}{4\pi\varepsilon_0}\frac{1}{r}\right]_{r_s}^{r_{st}}$$

$$\text{W} = \frac{q_s q_t}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{r_{st}}\right) = V_s - V_t = \text{potential difference}$$

If q_s and q_t are in coulombs and r and r_{st} in meters then potential difference is in volts.

2. (a) Consider resultant field due to two charges, q_s and q_t.

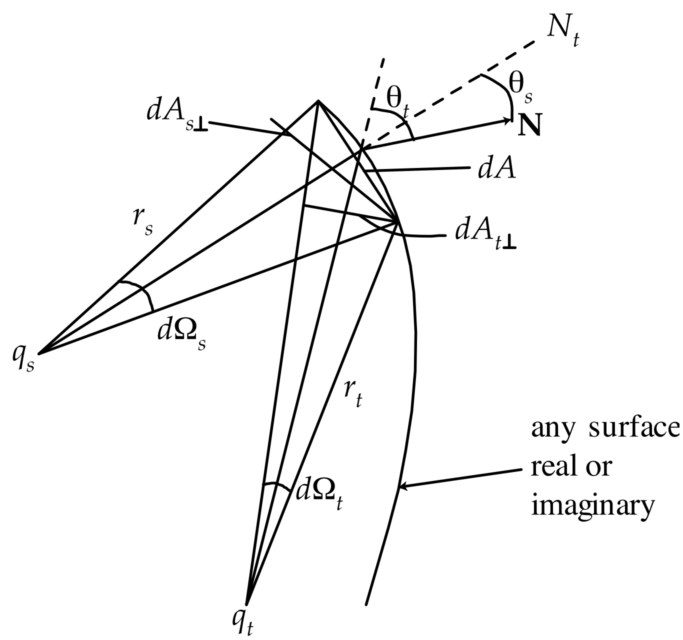

Fig. 22.3

$$
\begin{aligned}
df_s &= \frac{q_s d\Omega_s}{4\pi\varepsilon_0} \qquad d\Omega_s = \frac{dA_{s\perp}}{r_s^2} \qquad dA_{s\perp} = dA\cos\theta_s \\
df_t &= \frac{q_t d\Omega_t}{4\pi\varepsilon_0} \qquad d\Omega_t = \frac{dA_{t\perp}}{r_2^2} \qquad dA_{t\perp} = dA\cos\theta_t \\
f_s &= \frac{q_s}{\varepsilon_0} \qquad\qquad f_t = \frac{q_t}{\varepsilon_0} \\
&\therefore \quad f_s + f_t = \frac{1}{\varepsilon_0}(q_s + q_t)
\end{aligned}
$$

$f_s + f_t$ is the total flux crossing any surface surrounding charges q_s and q_t.

By Eq. (22.1)

$$
\begin{aligned}
df_s &= E_s dA_{s\perp} = E_s \cos\theta_s dA \\
df_t &= E_t dA_{t\perp} = E_t \cos\theta_t dA
\end{aligned}
$$

Adding

$$\begin{aligned} df_s + df_t &= (E_s \cos\theta_s + E_t \cos\theta_t)\, dA \\ \mathbf{E}_s + \mathbf{E}_t &= \mathbf{E}_R = \text{resultant field} \end{aligned} \tag{22.4}$$

Projection of $\mathbf{E}_s$ and $\mathbf{E}_t$ separately projected along normal $\mathbf{N}$ to the surface is the same as the projection of $\mathbf{E}$ along $\mathbf{N}$. That is,

$$\begin{aligned} \mathbf{E}_s \cdot \mathbf{N} + \mathbf{E}_t \cdot \mathbf{N} &= (\mathbf{E}_s + \mathbf{E}_t) \cdot \mathbf{N} = \mathbf{E}_R \cdot \mathbf{N} \\ \text{or} \qquad E_s \cos\theta_s + E_t \cos\theta_t &= E_R \cos\theta \end{aligned} \tag{22.5}$$

θ is the angle the resultant makes with $\mathbf{N}$ (not shown in the figure).

Let $\mathbf{E}_R = \mathbf{E}$

$$\begin{aligned} df_s + df_t &= E \cos\theta dA \\ \text{or} \qquad f_s + f_t &= \int E \cos\theta dA \\ \text{But} \qquad f_s + f_t &= \frac{1}{\varepsilon_0} (q_s + q_t) \end{aligned}$$

Therefore

$$\boxed{\int E \cos\theta dA = \frac{1}{\varepsilon_0} (q_s + q_t)} \qquad \text{Gauss Law} \tag{22.6}$$

Eq. (22.6) applies to any number of charges.

2. (b) Force Between Moving Charges. Magnetic Field

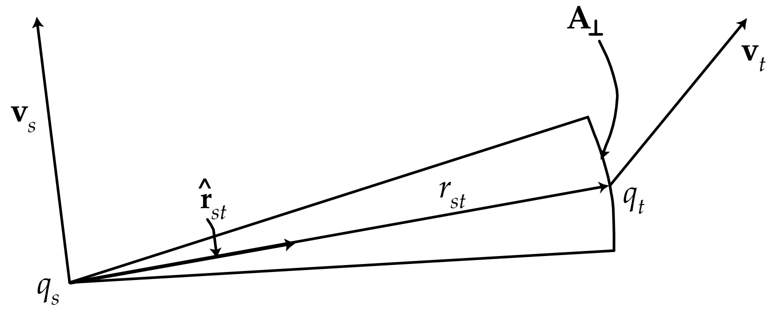

Fig. 22.4

Flux from q_s at position $\mathbf{r}_{st}$ is proportional to q_s, $d\Omega$ and some function of its velocity, $\mathbf{v}_s$, and direction, $\hat{\mathbf{r}}_{st}$. Thus

$$\begin{aligned} \text{Flux} &= d\mathbf{f} = \frac{\mu_0}{4\pi} q_s f\left(\mathbf{v}_s, \hat{\mathbf{r}}_{st}\right) d\Omega = \frac{\mu_0}{4\pi} q_s f\left(\mathbf{v}_s, \hat{\mathbf{r}}\right) \frac{dA_\perp}{r^2} \\ \frac{d\mathbf{f}}{dA_\perp} &= \frac{\mu_0 q_s}{4\pi} \frac{f\left(\mathbf{v}_s, \hat{\mathbf{r}}_{st}\right)}{r^2} \qquad \text{Call } \frac{d\mathbf{f}}{dA_\perp} = \mathbf{B} \end{aligned} \tag{22.7}$$

For positive charge $\mathbf{B} = \frac{\mu_0 q_s}{4\pi} \frac{\mathbf{v}_s \times \hat{\mathbf{r}}_{st}}{r^2}$ where by the right hand screw rule (right hand screw turning from $\mathbf{v}_s$ to $\hat{\mathbf{r}}_{st}$) $\mathbf{v}_s \times \hat{\mathbf{r}}_{st} = v_s \sin\theta_s \hat{\mathbf{r}}_{st}$

$$\begin{aligned} q_s \mathbf{v}_s &= q_s \frac{d\mathbf{l}_s}{dt} \\ q_s \frac{d\mathbf{l}_s}{dt} &= I_s \\ \therefore\ q_s \mathbf{v}_s &= I_s d\mathbf{l}_s \end{aligned}$$

$$\therefore \quad \mathbf{B} = \frac{\mu_0}{4\pi} \frac{I_s d\mathbf{l}_s \times \hat{\mathbf{r}}}{r^2} \qquad \text{Magnetic field. Biot-Savart Law} \tag{22.8}$$

$$\begin{aligned} d\mathbf{l}_s \times \hat{\mathbf{r}} &= d\ell_s \sin\theta_s \mathbf{n} \qquad \mu_0 = 4\pi \times 10^{-7} \frac{\text{Wb}}{\text{Am}} \\ \mathbf{n} &= \text{unit normal between } d\mathbf{l}_s \text{ and } \hat{\mathbf{r}}_{st} \end{aligned}$$

$\mathbf{B}$ exerts force on $q_t \mathbf{v}_t$. Unit of $\mathbf{B}$ is defined through the effect of $\mathbf{B}$ on $q_t \mathbf{v}_t$.

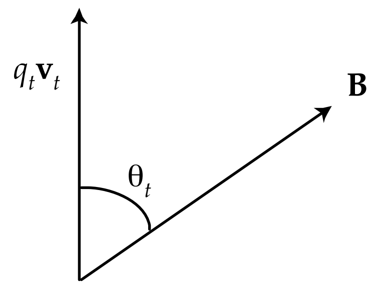

Fig. 22.5

$$\mathbf{F} = q_t\mathbf{v}_t \times \mathbf{B}$$

$$\mathbf{F} = q_t v B \sin\theta \mathbf{n}$$

or
$$\mathbf{F} = I_t d\ell_t B \sin\theta \mathbf{n}$$

When
$$\begin{aligned} \mathbf{F} &= 1 \text{ Newton}, \qquad I_t = 1 \text{ amp} \qquad \text{and} \qquad d\ell_t = 1 \text{ m} \\ \sin\theta &= 1 \qquad \text{then} \\ B &= 1 \text{ Tesla} \qquad 1 \text{ gauss} = 10^{-4} \text{ T} \end{aligned}$$

Biot-Savart Law
$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I_s d\mathbf{l}_s \times \hat{\mathbf{r}}}{r^2}$$

22.2 Magnetic Field at Distance a from a Long Wire Carrying a Current, I_s.

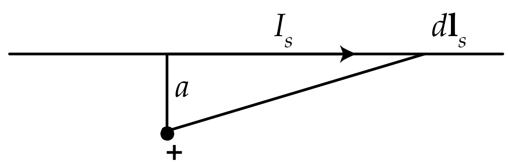

Fig. 22.6

$$\begin{aligned} B &= \frac{\mu_0 I_s}{2\pi a} = \frac{2\mu_0}{4\pi}\frac{I_s}{a} \\ \frac{\mu_0}{4\pi} &= 10^{-7}\frac{\text{T}\cdot\text{m}}{\text{Amp}} \\ \frac{1}{4\pi\varepsilon_0} &= 8.98755 \times 10^9 \frac{\text{N m}}{\text{coul}^2} \\ \frac{\frac{1}{4\pi\varepsilon_0}}{\frac{\mu_0}{4\pi}} &= \frac{1}{\mu_0\varepsilon_0} \equiv \frac{9\times 10^9}{10^{-7}} = 9\times 10^{16}\frac{\text{m}^2}{\text{sec}^2} = c^2 \end{aligned}$$

Force between two parallel wires: Definition of Ampere

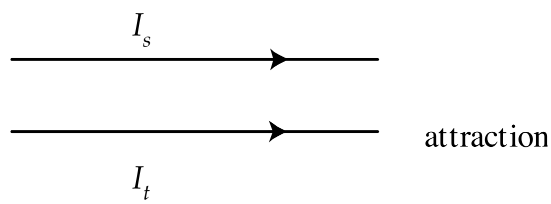

Fig. 22.7

$$
\begin{aligned}
B_s &= \frac{\mu_0 I_s}{2\pi a} \\
F &= B\ell_t I_t \\
\frac{F}{\ell_t} &= BI_t = \frac{\mu_0 I_s I_t}{2\pi a} \\
I &= 1 \text{ amp if} \qquad \frac{F}{\ell} = 2\times 10^{-7}\frac{\text{Newtons}}{\text{length}} \\
2\times 10^{-7} &= \frac{\mu_0}{2\pi}\frac{I\times I}{1} \\
\therefore & \quad \frac{\mu_0}{4\pi} \equiv 10^{-7} \\
\text{Coulomb} &= 1 \text{ ampere/ sec}
\end{aligned}
$$

– –

$$
\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi}\frac{Id\mathbf{l}\times\hat{\mathbf{r}}}{R^2} = \frac{\mu_0 I}{4\pi}\frac{2\pi R}{R^2} \\
B &= \frac{\mu_0 I}{2R} \qquad (22.9)
\end{aligned}
$$

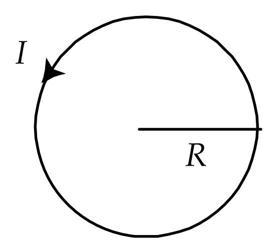

Fig. 22.8

Walk around wire

$$\begin{aligned} \mathbf{B} \cdot 2\pi R &= \frac{\mu_0 I}{2\pi a} 2\pi a \\ \int B \cos\theta d\ell &= \mu_0 I \end{aligned}$$

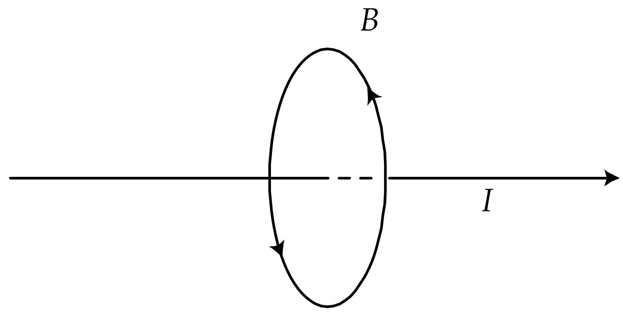

Fig. 22.9

Field in solenoid

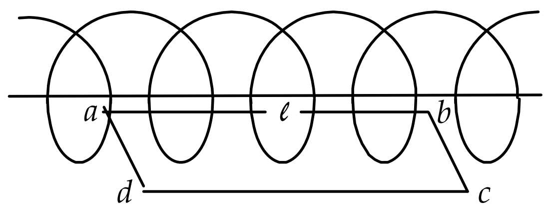

Fig. 22.10

$$\int \mathbf{B} \cdot d\mathbf{l} = B\ell_{ab} + o\ell_{bc} + o\ell_{cd} + o\ell_{da}$$

Let

$$\begin{aligned} \ell_{ab} &= \ell \\ B\ell &= \mu_0 N I \qquad N = \text{number of turns in length } \ell \\ B &= \mu_0 \frac{N}{\ell} I \end{aligned}$$

Faraday's law

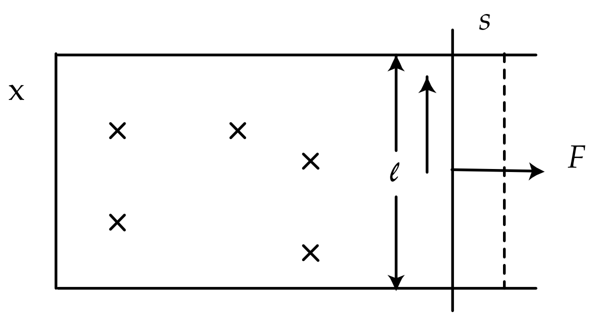

Fig. 22.11

$$\text{W}= Fs = Bvqs$$

$$\mathcal{E}mf = \mathcal{E} = \frac{\text{W}}{q} = Bvs$$

or

$$\begin{aligned} \text{W} &= Fs = BI\ell s \\ P &= \frac{\text{W}}{t} = BI\frac{(\ell s)}{t} = BI\left(\frac{\Delta A}{\Delta t}\right) \\ P &= \mathcal{E}I \qquad\qquad \text{W} = \mathcal{E}q, \\ \therefore\ \mathcal{E}I &= -BI\frac{\Delta A}{\Delta t} \qquad\qquad P = \frac{\text{W}}{t} = \mathcal{E}\frac{q}{t} = \mathcal{E}I \\ \mathcal{E} &= -B\frac{\Delta A}{\Delta t} \end{aligned}$$

In general

$$\boxed{\mathcal{E} = -\frac{\Delta (BA)}{\Delta t} = -\frac{\Delta \phi}{\Delta t}} \quad \text{Faraday's Law} \tag{22.10}$$

22.3 Kepler's Three Laws of Planetary Motion

Kepler's three laws of planetary motion are as follows:

1) The orbit of a planet forms an ellipse with the Sun at one focus.
2) The Sun-planet radius vector sweeps out equal areas in equal times.
3) The square of the period of revolution of a planet is proportional to the cube of the semimajor axis of its elliptical orbit.

We derive Kepler's Third Law since it is simple to do. The three laws are derived in books on astronomy or physics. A. Douglas Davis, (Davis 1986), gives a nice treatment. See also A.P. French (1971) and Frank H. Shu (1982).

Assuming that the only law of gravitation between two masses M_s and M_t separated by a distance R is given by the reversible equation:

$$F_{ts} = \frac{GM_s m_t}{R^2} \tag{22.11}$$

one can say

$$\begin{aligned} \frac{GM_s m_t}{R^2} &= \frac{m_t v_t^2}{R} \\ \frac{GM_s}{R} &= v_t^2 \end{aligned} \tag{22.12}$$

The value of v_t may be regarded as an initial condition that determines the value of R for a given value of v_t in the solar sytem or for any value of M_s.

$$M_s = \frac{v_t^2 R}{G} \tag{22.13}$$

Use Eq. (22.12) to obtain Kepler's Third Law by introducing the period

$$P_t = \frac{2\pi R}{v_t}$$

or

$$v_t = \frac{2\pi R}{P_t}$$

Substituting in Eq. (22.12)

$$\frac{GM}{R^2} = \left(\frac{2\pi R}{P_t}\right)^2 \quad \rightarrow \quad \frac{GM_s}{(4\pi)^2} = \frac{R^3}{P_t^2}$$

So that

$$\begin{aligned} P_t^2 &= \frac{(4\pi)^2 R^3}{GM_s} \\ \frac{P_t^2}{R^3} &= \frac{(4\pi)^2}{GM_s} = \text{a constant} \end{aligned}$$