19

PARITY. DIRECT PRODUCT OF TWO TRIVECTORS.

19.1 Electro-Weak "Force": 3D Formulation of the Biot-Savart Law

In this Chapter, we show the 3D formulation of the Biot-Savart Law by substituting associative products in place of dot and cross products. It is shown that the electroweak force is a trivector. The word "force" in this phrase is misleading as no physical three dimensional force occurs; only a "trivector." The term parity occurs in the following dicussions. It is defined as fallows. If a quantity or an experiment is invariant when projected through the origin if is said to satisfy parity. If it changes sign, then it violates parity. For example $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ violates partity

The Biot-Savart law specifying the magnetic field at a distance r from source current $q_s\mathbf{v}_s$ is

$$\mathbf{B}_s = \frac{\mu_0 q_s \mathbf{v}_s \times \mathbf{r}}{4\pi r^3} = \mu_0 k \mathbf{v}_s \times \mathbf{r} = \mu_0 k \left(\mathbf{i}B_{sx} + \mathbf{j}B_{sy} + \mathbf{k}B_{sz}\right) \qquad k = 1/4\pi r^3 \tag{19.1}$$

The force on a test current $q_t\mathbf{v}_t$ is given by Faraday's law

$$\mathbf{F}_{ts} = q_t\mathbf{v}_t \times \mathbf{B}_s = \mu_0 k q_t q_s \mathbf{v}_t \times \left(\mathbf{v}_s \times \mathbf{r}_{st}\right) \tag{19.2}$$

$\mathbf{r}_{st}$ is the distance from element of current $q_s\mathbf{v}_s$ to element of current $q_t\mathbf{v}_t$.

Now express Eq. (19.1) in 3-dimensional Clifford algebra by replacing the cross product $\mathbf{v}_s \times \mathbf{r}$ by the associative product $\mathbf{v}_s\mathbf{r}$. Normally we would simply replace $\mathbf{v}_s \times \mathbf{r}$ by the wedge product $\mathbf{v}_s \wedge \mathbf{r} \equiv \left(\mathbf{v}_s\mathbf{r} - \mathbf{r}\mathbf{v}_s\right)/2$. Our motivation in replacing the cross product by the associative product is to explore the possibility that additional terms may yield new interactions of physical significance. This is in line with our hypothesis that all physics is contained in the information and constraints provided by space-time algebra. In this initial exercise we limit ourselves to 3 dimensional Clifford algebra. Next we will write all relationships in space-time algebra, that is, time will be included as the fourth dimension. Replacing $\mathbf{v}_s \times \mathbf{r}$ in Eq. (19.1) by

$$\begin{aligned} \mathbf{v}_s\mathbf{r} &= (\mathbf{v}_s\mathbf{r} + \mathbf{r}\mathbf{v}_s)/2 + (\mathbf{v}_s\mathbf{r} - \mathbf{r}\mathbf{v}_s)/2 = \mathbf{v}_s \bullet \mathbf{r} + \mathbf{v}_s \wedge \mathbf{r} \qquad (19.3) \\ &= (v_{sx}x + v_{sy}y + v_{sz}z) \\ &\quad + (yv_x - xv_y)\,\mathbf{e}_1\mathbf{e}_2 + (xv_z - zv_x)\,\mathbf{e}_3\mathbf{e}_1 + (zv_y - yv_z)\,\mathbf{e}_2\mathbf{e}_3 \\ &= \mathbf{v} \bullet \mathbf{r} + B_{sz}\mathbf{e}_1\mathbf{e}_2 + B_{sy}\mathbf{e}_3\mathbf{e}_1 + B_{sx}\mathbf{e}_2\mathbf{e}_3 \qquad (19.4) \end{aligned}$$

Now multiply the bivector in (19.4) by $(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)(-\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) = 1$ to obtain:

$$\mathbf{v}_s\mathbf{r} = (\mathbf{v} \bullet \mathbf{r}) + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\left[(zv_x - xv_y)\,\mathbf{e}_3 + (xv_z - zv_x)\,\mathbf{e}_2 + (zv_y - yv_x)\,\mathbf{e}_1\right]$$

or in partial Gibbs notation

$$\begin{aligned} \mathbf{v}_s\mathbf{r} &= (\mathbf{v} \bullet \mathbf{r}) + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\left[B_{sz}\mathbf{e}_3 + B_{sy}\mathbf{e}_2 + B_{sx}\mathbf{e}_1\right] \\ &= (\mathbf{v}_s \bullet \mathbf{r}) + (\mathbf{v}_s \wedge \mathbf{r}) = (\mathbf{v}_s \cdot \mathbf{r}) + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\,(\mathbf{v}_s \times \mathbf{r}) \\ &= (\mathbf{v}_s \bullet \mathbf{r}) + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{B}_s \qquad (19.5) \end{aligned}$$

$$\mathbf{B}_s = \mathbf{v}_s \times \mathbf{r} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_{sx} & v_{sy} & v_{sz} \\ x & y & z \end{vmatrix} = B_{sx}\mathbf{e}_1 + B_{sy}\mathbf{e}_2 + B_{sz}\mathbf{e}_3$$

The direct or associative product of two pure multivectors **a** and **b** can always be written

$$\mathbf{ab} = (\mathbf{ab} + \mathbf{ba})/2 + (\mathbf{ab} - \mathbf{ba})/2 \qquad (19.6)$$

The first term is called the symmetric or inner or dot product. the second term is called the anti-symmetric or outer or wedge product.

The rank or blade of a pure multivector refers to the number of unit vectors composing it. Eq. (19.3) is the direct product of two pure vectors. The symmetric product in this case is the lower rank object (a scalar) and the wedge product is a bivector. We define the dot product in general as the term in (19.3) that yields the lower rank multivector. When **a** and **b** are vectors this is the combination $(\mathbf{ab} + \mathbf{ba})/2$ in Eq. (19.3) since it is a scalar. The terms $(\mathbf{ab} - \mathbf{ba})/2$ in this case is a bivector. Eq. (19.10) is the direct product of a bivector and a vector. In this case the symmetric sum is a trivector and the antisymmetric sum is a vector, a lower rank object. However, we now designate antisymmetric sum by the dot so that $\mathbf{B} \bullet \mathbf{v} = (\mathbf{Bv} - \mathbf{vB})/2$. This means that we have to know which combination in (19.6) is the lower rank before using the dot notation. A formula can be written giving that specifies the combination of wedge and dot products of multivectors that will result in a higher or lower rank. However, in the present case, we deal only with four pure multivectors $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ so it is a simple matter to simply evaluate the direct product of the 2 multivectors if interest and then, by inspection, assign the dot product notation $\mathbf{a} \bullet \mathbf{b}$ to whichever combination $(\mathbf{ab} + \mathbf{ba})/2$ or $(\mathbf{ab} - \mathbf{ba})/2$ is the lower rank combination. Greider adopted this notation and we follow it here by force of habit. It is a kind of hangover from Gibbs notation where $\mathbf{a} \cdot \mathbf{b}$ is a scalar and the cross product $\mathbf{a} \times \mathbf{b}$ is a vector. One can eliminate possible confusion by not introducing the dot, $\bullet$, and wedge notation, but simply write $(\mathbf{ab} + \mathbf{ba})/2$, $(\mathbf{ab} - \mathbf{ba})/2$ and assign the dot or wedge notation for these sums. Greider's student, John Ross, always defined the combination $(\mathbf{ab} + \mathbf{ba})/2$ as the dot product $\mathbf{a} \bullet \mathbf{b}$, and $(\mathbf{ab} - \mathbf{ba})/2$ as the wedge product $\mathbf{a} \wedge \mathbf{b}$. This is a reasonable notation but the dot product then is not always the lower rank combination.

$\mu_0 k q_s (\mathbf{v}_s \times \mathbf{r})$ in Eq. (19.1) has been replaced by

$$\mu_0 k q_s (\mathbf{v}_s \mathbf{r}) = \mu_0 k q_s (\mathbf{v}_s \bullet \mathbf{r} + \mathbf{v}_s \wedge \mathbf{r}) = \mu_0 k q_s [\mathbf{v}_s \cdot \mathbf{r} + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 (\mathbf{v}_s \times \mathbf{r})] \tag{19.7}$$

The right hand side has been expressed in a mixed notation since we have written $\mathbf{v}_s \times \mathbf{r} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 (\mathbf{v}_s \times \mathbf{r})$ and $\mathbf{v}_s \cdot \mathbf{r} = \mathbf{v}_s \bullet \mathbf{r}$.

Now further generalize the force equation Eq. (19.2) by replacing $\mathbf{v}_t \times (\mathbf{v}_s \times \mathbf{r})$ by the associative product $(\mathbf{v}_s\mathbf{r})\mathbf{v}_t$. That is, the first associative product $(\mathbf{v}_s\mathbf{r})$ is followed by its associative product with $\mathbf{v}_t$. The order of $\mathbf{v}_t$ and $(\mathbf{v}_s\mathbf{r})$ is chosen to give a positive sign later. We then conjecture that the force of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$ is

$$\mathbf{F}_{ts} = \mu_0 k q_t q_s [(\mathbf{v}_s \bullet \mathbf{r})\mathbf{v}_t + (\mathbf{v}_s \wedge \mathbf{r})\mathbf{v}_t] \tag{19.8}$$

where

$$\mathbf{v}_s \wedge \mathbf{r} = B_{sz}\mathbf{e}_1\mathbf{e}_2 + B_{sy}\mathbf{e}_3\mathbf{e}_1 + B_{sx}\mathbf{e}_2\mathbf{e}_3 \tag{19.9}$$

$$\mathbf{v}_s \bullet \mathbf{r} = v_{sx}x + v_{sy}y + v_{sz}z$$

Set the term $(\mathbf{v}_s \bullet \mathbf{r})\mathbf{v}_t$ aside for the moment and examine

$$(\mathbf{v}_s \wedge \mathbf{r})\mathbf{v}_t = \mathbf{B}_s\mathbf{v}_t \equiv \left(\overset{\text{Vec}}{\mathbf{B}_s\mathbf{v}_t - \mathbf{v}_t\mathbf{B}_s}\right)/2 + \left(\overset{\text{Triv}}{\mathbf{B}_s\mathbf{v}_t + \mathbf{v}_t\mathbf{B}_s}\right)/2 \tag{19.10}$$

$$\mathbf{B}_s\mathbf{v}_t = \mathbf{B}_s \bullet \mathbf{v}_t + \mathbf{B}_s \wedge \mathbf{v}_t \tag{19.11}$$

$$\begin{aligned} &= [(v_{ty}B_{sz} - v_{tz}B_{sy})\mathbf{e}_1 + (v_{tz}B_{sx} - v_{tx}B_{sz})\mathbf{e}_2 + (v_{tx}B_{sy} - v_{ty}B_{sx})\mathbf{e}_3] \\ &\quad + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 (v_{tx}B_x + v_{ty}B_y + v_{tz}B_z) \end{aligned} \tag{19.12}$$

$$= \mathbf{B}_s \bullet \mathbf{v}_t + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 (\mathbf{v}_t \cdot \mathbf{B}_s)$$

$$= (\mathbf{v}_t \times \mathbf{B}_s) + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 (v_{tx}B_{sx} + v_{ty}B_{sy} + v_{tz}B_{sz})$$

The vector term $(\mathbf{B}_s \bullet \mathbf{v}_t)$ in Eq. (19.11) has the same sign as $\mathbf{v}_t \times \mathbf{B}_s$ in the vector expression for the force $\mathbf{F}_{ts}$ of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$, Eq. (19.2). Therefore the magnetic force of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$ in Clifford algebra is given by

$$\mathbf{F}_{ts} = (\mathbf{v}_s \wedge \mathbf{r}) \bullet \mathbf{v}_t = \mathbf{B}_s \bullet \mathbf{v}_t = -\mathbf{v}_t \bullet \mathbf{B}_s$$

Going back to Eq. (19.8) we then have (note: $\mathbf{v}_t \wedge \mathbf{B}_s \equiv \mathbf{B}_s \wedge \mathbf{v}_t$)

$$\mathbf{F}_{ts} = \mu_0 k q_t q_s \;[\overset{\text{I}}{\underset{\substack{\downarrow \\ \text{new term} \\ \text{Scalar field} \\ \text{acting on Vec}}}{\mathbf{v}_t(\mathbf{v}_s \bullet \mathbf{r})}} + \overset{\text{II}}{\underset{\substack{\downarrow \\ \text{Faraday} \\ \text{vector force}}}{\mathbf{B}_s \bullet \mathbf{v}_t}} + \overset{\text{III}}{\underset{\substack{\downarrow \\ \text{new term} \\ \text{(trivector)}}}{\mathbf{B}_s \wedge \mathbf{v}_t}}] \tag{19.13}$$

Details:

$$\begin{aligned}
\mathbf{B}_s\mathbf{v}_t &= (B_{sz}\mathbf{e}_1\mathbf{e}_2 + B_{sy}\mathbf{e}_3\mathbf{e}_1 + B_{sx}\mathbf{e}_2\mathbf{e}_3)(v_{tx}\mathbf{e}_1 + v_{ty}\mathbf{e}_2 + v_{tz}\mathbf{e}_3) \\
&= -v_{tx}B_{sz}\mathbf{e}_2 + v_{ty}B_{sz}\mathbf{e}_1 + v_{tz}B_{sz}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\
&\quad + v_{tx}B_{sy}\mathbf{e}_3 + v_{ty}B_{sy}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 - v_{tz}B_{sy}\mathbf{e}_1 \\
&\quad + v_{tx}B_{sx}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 - v_{ty}B_{sx}\mathbf{e}_3 + v_{tz}B_{sx}\mathbf{e}_2 \\
&= (v_{ty}B_{sz} - v_{tz}B_{sy})\mathbf{e}_1 + (v_{tz}B_{sx} - v_{tx}B_{sz})\mathbf{e}_2 + (v_{tx}B_{sy} - v_{ty}B_{sx})\mathbf{e}_3 \\
&\quad + (v_{tx}B_{sx} + v_{ty}B_{sy} + v_{tz}B_{sz})\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\
\mathbf{v}_t\mathbf{B}_s &= (v_{tx}\mathbf{e}_1 + v_{ty}\mathbf{e}_2 + v_{tz}\mathbf{e}_3)(B_{sz}\mathbf{e}_1\mathbf{e}_2 + B_{sy}\mathbf{e}_3\mathbf{e}_1 + B_{sx}\mathbf{e}_2\mathbf{e}_3) \\
&\quad v_{tx}B_{sz}\mathbf{e}_2 - v_{tx}B_{sy}\mathbf{e}_3 + v_{tx}B_{sx}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\
&\quad - v_{ty}B_{sz}\mathbf{e}_1 + v_{ty}B_{sy}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + v_{ty}B_{sx}\mathbf{e}_3 \\
&\quad + v_{tz}B_{sz}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + v_{tz}B_{sy}\mathbf{e}_1 - v_{tz}B_{sx}\mathbf{e}_2
\end{aligned}$$

$$\begin{aligned}
&= (v_{tz}B_{sy} - v_{ty}B_{sz})\mathbf{e}_1 + (v_{tx}B_{sz} - v_2B_{sx})\mathbf{e}_2 + (v_{ty}B_{sx}v_{tx}B_{sy})\mathbf{e}_3 \\
&\quad + (v_{tx}B_{sx} + v_{ty}B_{sy} + v_{tz}B_{sz})\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3
\end{aligned}$$

add

$$(\mathbf{B}_s\mathbf{v}_t + \mathbf{v}_t\mathbf{B}_s) = 2[(v_{tx}B_{sx} + v_{ty}B_{sy} + v_{tz}B_{sz})\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3] = 2\mathbf{B}_s \wedge \mathbf{v}_t$$

subtract

$$\begin{aligned}
(\mathbf{B}_s\mathbf{v}_t - \mathbf{v}_t\mathbf{B}_s) &= 2\left[(v_{ty}B_{sz} - v_{tz}B_{sy})\mathbf{e}_1 + (v_{tz}B_{sx} - v_{tx}B_{sz})\mathbf{e}_2\right. \\
&\quad \left. + (v_{tx}B_{sy} - v_{ty}B_{sx})\mathbf{e}_3\right] = 2(\mathbf{B}_s \bullet \mathbf{v}_t) = 2(\mathbf{v}_t \times \mathbf{B}_s)
\end{aligned}$$

Thus, in summary

$$\begin{aligned} (\mathbf{B}_s\mathbf{v}_t + \mathbf{v}_t\mathbf{B}_s)/2 &= (\mathbf{B}_s \wedge \mathbf{v}_t) = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3(\mathbf{v}_t \cdot \mathbf{B}_s) = \text{triv} \qquad (19.14) \\ (\mathbf{B}_s\mathbf{v}_t - \mathbf{v}_t\mathbf{B}_s)/2 &= (\mathbf{B}_s \bullet \mathbf{v}_t) = (\mathbf{v}_t \times \mathbf{B}_s) = \text{vec} \qquad (19.15) \end{aligned}$$

Eqs. (19.13) and (19.15) express the result in both Clifford algebra and Gibbs notation.

Note that for all pairs of multivectors $\mathbf{B}_s \wedge \mathbf{v}_t \equiv \mathbf{v}_t \wedge \mathbf{B}_s$

Since

$$\begin{aligned} \mathbf{v}_t \times \mathbf{B}_s &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_{tx} & v_{ty} & v_{tz} \\ B_{sx} & B_{sy} & B_{sz} \end{vmatrix} \\ &= (v_{ty}B_{sz} - v_{tz}B_{sy})\mathbf{e}_1 + (v_{tz}B_{sx} - v_{tx}B_{sz})\mathbf{e}_2 + (v_{tx}B_{sy} - v_{tz}B_{sx})\mathbf{e}_3 \end{aligned}$$

$\mathbf{B}_s\mathbf{v}_t$ may be written in the natural Clifford notation or in the usual Gibbs notation.

$$\mathbf{B}_s\mathbf{v}_t = \overset{\text{Cl. Alg.}}{\mathbf{B}_s \bullet \mathbf{v}_t + \mathbf{B}_s \wedge \mathbf{v}_t} = \overset{\text{Gibbs}}{\mathbf{v}_t \times \mathbf{B}_s + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3(\mathbf{v}_t \cdot \mathbf{B}_s)}$$

The term $\mu_0 k q_s q_t \mathbf{B}_s \bullet \mathbf{v}_t$ gives the Faraday law in Clifford notation. Terms I and III in (19.13) are new. Consider the new term $\mathbf{B}_s \wedge \mathbf{v}_t$ in Eq. (19.13).

$$\mu_0 k q_t \mathbf{B}_s \wedge \mathbf{v}_t = \mu_0 k q_t \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3(v_{tx}B_{sx} + v_{ty}B_{sy} + v_{tz}B_{sz}) = \mu_0 k q_t \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3(\mathbf{v}_t \cdot \mathbf{B}_s) \qquad (19.16)$$

$q_t\mathbf{v}_t$ may be replaced by current density $\mathbf{j}_t$ or momentum $\mathbf{p}_e$. Thus

$$q_t\mathbf{v}_t \rightarrow \mathbf{p}_e$$

$\mu_0 k \mathbf{B}_s$ may be replaced by the nuclear angular momentum $\mathbf{r} \wedge \mathbf{p}_e = \mathbf{L}_n$, since it is a bivector. Then

$$\mu_0 k \mathbf{B}_s \rightarrow \mathbf{J}_n = \text{nuclear spin}$$

and (19.16) could then be replaced by

$$q_t\mathbf{v}_t \wedge \mu_0 k \mathbf{B}_s \rightarrow \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\,\mathbf{p}_e \cdot \mathbf{J}_n = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3(p_{ex}J_{nx} + p_{ey}J_{ny} + p_{ez}J_{nz}) \qquad (19.17)$$

Eq. (19.17) is a trivector, $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ multiplied by a scalar $s = p_{ex}J_{nx} + p_{ey}J_{ny} + p_{ez}J_{nz}$. That is, one has a quantity $\mathbf{s}' = s\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. $\mathbf{s}'$ is called a pseudo scalar (or trivector) since $\mathbf{s}'^2 = -1$. The quantity $\mathbf{s} = s'\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ does not remain invariant under inversion: when $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are replaced by their negatives, it changes sign. Therefore it violates parity conservation. Parity conservation means that the equation describing the interaction is invariant under inversion. Note that in space-time algebra formulation we do not need to look at the coefficient $p_{ex}J_{nx} + p_{ey}J_{ny} + p_{ez}J_{nz}$. It is simply a scalar.

Lee and Yang, however, examined the quantity $p_{ex}J_{nx}+p_{ey}J_{ny}+p_{ez}J_{nz}$ which they write as $\mathbf{p}_e \cdot \mathbf{J}_n$. Note that the corresponding quantity in Clifford algebra formalism is

$$\mathbf{p}_e \wedge \mathbf{J}_n = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\,(p_{ex}J_{nx} + p_{ey}J_{ny} + p_{ez}J_{nz}) \tag{19.18}$$

The right hand side of (19.18) is a trivector times a scalar.

In (19.18) and (19.19) $\mathbf{J}_n$ is nuclear spin, a bivector.

$$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\,\mathbf{p}_e \cdot \mathbf{J}_n \tag{19.19}$$

The dot now denotes the Gibbs dot product. Note that in Clifford notation

$$\mathbf{p}_e \cdot \mathbf{J_n} \neq \mathbf{p}_e \bullet \mathbf{J}_n$$

The Clifford algebra dot product is only equal to the Gibbs dot product when two vectors are involved. That is, $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \bullet \mathbf{b}$ only when $\mathbf{a}$ and $\mathbf{b}$ are vectors. In equation (19.17) $\mathbf{p}_t$ is a vector and $\mathbf{J}_s$ is a bivector (axial vector $(q_s\mathbf{r} \times \mathbf{v}_s)$). If we work only with the expression $p_{ex}J_{nx} + p_{ey}J_{ny} + peJ_{nz}$ and write the interaction, F_{en} in Gibbs notation as

$$\mathbf{F}_{en} = \mathbf{p}_e \cdot \mathbf{J}_n \tag{19.20}$$

then we have to look at the individual terms $\mathbf{p}_e$ and $\mathbf{J}_n$in (19.20) under inversion as done in Burcham and Jobes. It is interesting to do this anyway since it gives a clear physical picture of what happens to $\mathbf{p}_e$ and $\mathbf{J}_n$ under inversion. One can see that $\mathbf{p}_e \cdot \mathbf{J}_n$ changes sign under inversion since $\mathbf{J}$ is an axial vector $\mathbf{r} \times \mathbf{p}$ that does not change sign under inversion, however $\mathbf{p}_t$ is a polar vector, that is, a directed line segment vector, and does change sign. Thus the dot product of $\mathbf{p}$ and $\mathbf{J}$ does change sign under inversion. This interaction may be realized in β decay where the parent nucleus has spin $\mathbf{J}_n$, an axial vector, and $\mathbf{p}_e$ is a polar vector, the linear momentum of the emitted electron. An asymmetry in the angular distribution of the electrons measured with respect to the nuclear spin direction, that is, a net value of $\mathbf{p}_e \cdot \mathbf{J}_n$ in that direction would constitute proof of lack of invariance under parity, that is, under inversion of the experiment through the origin.

Fig 19.1 shows experimental results as described by Burchan and Jobes. They do not discuss the results. Fig. 1(a) shows electrons emitted in the direction of the momentum polar vector $\mathbf{p}_t$. Fig. 1(b) is the mirror reflection. Fig. 1(c) is the rotation of Fig. 1(b) by 180°. These two operations are equivalent to inverting the experiment through the origin. Imagine the inverted experimental result depicted by Fig. 1(c) to be placed next to the original experiment Fig 1(a). It is clear that they do not give the same result since the polar vector (electron momentum) has changed sign while the axial vector (nuclear spin) has not. In the inverted system the electrons are emitted preferentially along the general direction of the nuclear spin while they are emitted opposite to $\mathbf{J_n}$, the direction of the nucleus spin in the original experiment. The only way the experiment would satisfy parity invariance is for electrons to be emitted in equal amounts in both directions in the real experiment. Thus the fact that they are not emitted in equal amounts proves parity violation can exist in nature.

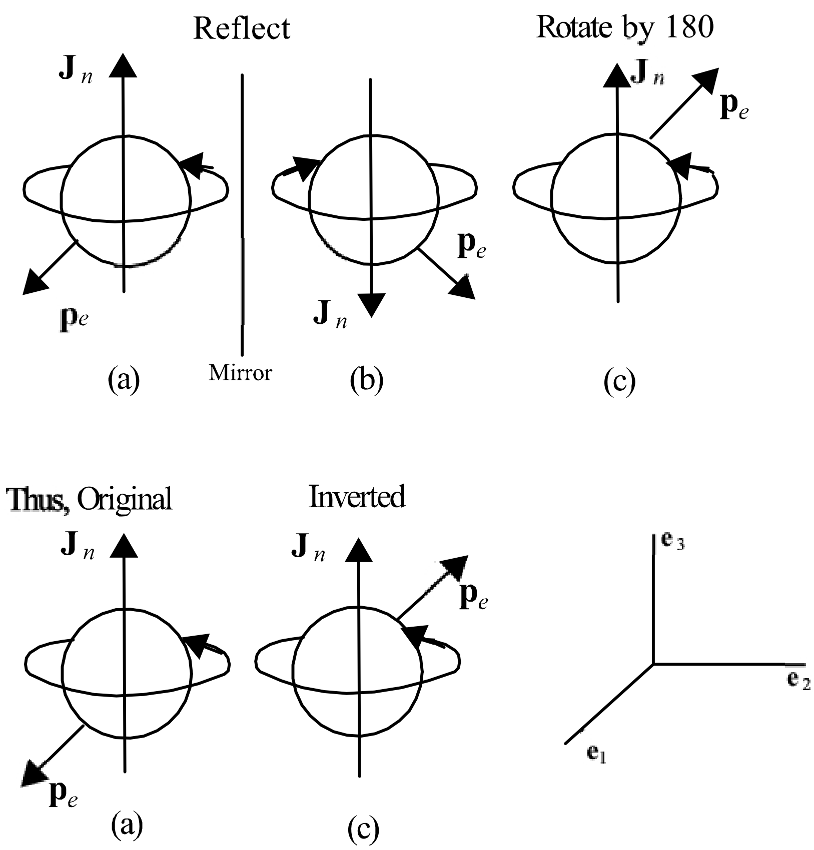

Fig 19.1

This result can also be seen in a different way by referring to the trivector terms in Eqs. (19.11) or (19.15). The interaction is

$$\mathbf{B}_s \wedge \mathbf{v}_t = (v_{xt}B_{sx} + v_{ty}B_{sy} + v_{tz}B_{sz})\,\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$$

Let $\mathbf{B}$ have the single component B_{sz}. The interaction must then have the form

$$\mathbf{F}_{st} = (v_{tz} B_{sz})\, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \to B_{sz} p_{tz} \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \qquad v_{tz} \sim p_{tz} \tag{19.21}$$

By Eq. (19.22) $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is a bivector plus a scalar. Therefore, the interaction described by Yang and Lee (Burchan and Jobes, Particle and Nuclear Physics) is not a weak "force"; it is a bivector. A bivector can represent a torque or angular momentum, for example.

Some repetitive comments plus Yukawa force:

$B_{sz} p_{tz}$ is a scalar times $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. The latter is a pseudo scalar since $(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) = -1$. If we identify $\mathbf{B}_{sz}$ with $\mathbf{e}_1\mathbf{e}_2$, then $B_{sz}\mathbf{e}_1\mathbf{e}_2$ corresponds to a directed plane and does not change its direction when $\mathbf{e}_1 \to -\mathbf{e}_1,\ \mathbf{e}_2 \to -\mathbf{e}_2$. However $\mathbf{p}_t$ is a vector which we identify with $\mathbf{e}_3$ and changes sign when $\mathbf{e}_3 \to -\mathbf{e}_3$. Thus, under inversion

$$(B_{sz}\mathbf{e}_1\mathbf{e}_2)(p_t\mathbf{e}_3) \to (B_{sz}\mathbf{e}_1\mathbf{e}_2)(-p_t\mathbf{e}_3)$$

That is, under inversion B_{sz} does not change sign, but $p_t\mathbf{e}_3$ does change sign. Thus under inversion the original configuration (a) changes to configuration (c) which is different from (a). The emission of electrons in a preferred direction in (a) is reversed under inversion, (c).

This experiment tells us that the new interaction term $\mathbf{B}_s \wedge \mathbf{v}_t$ in Eq. (19.13) does exist in nature and has been predicted by our generalization of the Biot-Savart and Faraday laws.

When using the Clifford algebra formulation we can see immediately that $(\mathbf{p}_e \cdot \mathbf{J}_n)\, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ $(\equiv \mathbf{J}_n \wedge \mathbf{v}_t)$ is a pseudo scalar since $(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) = -1$. That is, we need only examine the unit vectors in the trivector since the coefficient, a scalar, does not change sign. It is clear that a sign change occurs under inversion since $(-\mathbf{e}_1)(-\mathbf{e}_2)(-\mathbf{e}_3) = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$.

In the Gibbs form we have only the quantity $\mathbf{p}_e \cdot \mathbf{J}_n$. $\mathbf{J}_n$, an axial vector, which does not change sign under inversion. The vector $\mathbf{p}_e$, however, does change sign. Therefore we say that the net quantity $\mathbf{p}_e \cdot \mathbf{J}_n$ changes sign under inversion.

It is interesting to note the change that would apply after an interaction involving massive particles. We began by defining the spacial dependence of the force for a massless field through the geodesic spreading factor $k = 1/r^2$ and a corresponding potential $-1/r$ so that $\dfrac{d}{dt}\left(-\dfrac{1}{r}\right) = \dfrac{1}{r^2}$. However, for massive particle exchange in an interaction this potential must be multiplied by the Yukawa factor $e^{-\mu r}$ to obtain the Yukawa force, $-\frac{1}{4\pi}\frac{d}{dr}\left(\frac{e^{-\mu r}}{r}\right)$. There is nothing in the previous formalism to tell us which force law applies.

Differentiation of the Yukawa potential:

$$\begin{aligned} F &= -\frac{d}{dr}\left(\frac{e^{-\mu r}}{r}\right) = \frac{e^{-\mu r}}{r^2} - \frac{1}{r}\frac{d}{dr}e^{-\mu r} \\ &= \frac{e^{-\mu r}}{r^2} - \frac{e^{-\mu r}}{r}\frac{d}{dr}(-\mu r) \\ &= \frac{e^{-\mu r}}{r^2} + \frac{\mu e^{-\mu r}}{r} \end{aligned}$$

Thus close in $F = \frac{e^{-\mu r}}{r^2}$

Farther out $F = \frac{\mu e^{-\mu r}}{r}$

The first term in Eq. (19.8), $\mathbf{v}_t(\mathbf{v}_s \cdot \mathbf{r}_{st})$, indicates that there may be an interaction of the form

$$(v_{tx}\mathbf{e}_1 + v_{ty}\mathbf{e}_2 + v_{tz}\mathbf{e}_3)\left[v_{sx}(x_t - x_s) + v_{sy}(y_t - y_s) + v_{sz}(z_t - z_s)\right]$$

If real, this defines a scalar field $[\mathbf{v}_s \cdot (\mathbf{x}_t - \mathbf{x}_s)]$ interacting with a vector current $\mathbf{v}_t$ to give a vector force $\mathbf{v}_t[\mathbf{v}_s \cdot (\mathbf{x}_t - \mathbf{x}_s)]$. The above 3-D equations are generalized to space-time.

Burchan and Jobes, Particle and Nuclear Physics, p. 369 state that Lee and Yang pointed out that no experiment had been performed that was sensitive to parity violating effects and that in order to detect parity violation one would need to observe a pseudoscalar quantity. In β-decay one suitable pseudoscalar is
$\mathbf{J}_t \cdot \mathbf{p}_s$. A pseudoscalar as an observable which is invariant under rotation but which changes sign under spatial inversion (the parity operation); the scalar product of a polar vector and an axial vector is an example of a pseudoscalar. A scalar quantity on the other hand is invariant under both rotation and spatial inversion. Note that in the Clifford algebra formulation its square is also -1. The square of the Gibbs formula, $\mathbf{J}_t \cdot \mathbf{p}_s$, is $+1$.

The wedge product of the electromagnetic field bivector, $\mathbf{F}$, with velocity vector $\mathbf{V}$ is

$$\begin{aligned} \mathbf{F} &= \frac{E_x}{c}\mathbf{e}_1 + \frac{E_y}{c}\mathbf{e}_2 + \frac{E_z}{c}\mathbf{e}_3 + B_z\mathbf{e}_1\mathbf{e}_2 + B_y\mathbf{e}_3\mathbf{e}_1 + B_x\mathbf{e}_2\mathbf{e}_3 \\ \mathbf{V} &= v_0\mathbf{e}_0 + v_x\mathbf{e}_1 + v_y\mathbf{e}_2 + v_z\mathbf{e}_3, \qquad v_0 = c \end{aligned}$$

$$\begin{aligned}
\mathbf{F}\wedge\mathbf{V} &= (\mathbf{FV}+\mathbf{VF})/2 \\
&= \mathbf{e}_5\left\{(v_xB_x+v_yB_y+v_zB_z)\,\mathbf{e}_0+\left(v_0B_x-v_y\frac{E_z}{c}+v_z\frac{E_y}{c}\right)\mathbf{e}_1\right. \\
&\quad \left.+\left(v_0B_y+v_x\frac{E_z}{c}-v_z\frac{E_x}{c}\right)\mathbf{e}_2+\left(v_0B_z-v_x\frac{E_y}{c}+v_y\frac{E_z}{c}\right)\mathbf{e}_3\right\} \\
&= \mathbf{e}_5\left\{(\mathbf{v}\cdot\mathbf{B})\,\mathbf{e}_0+\left[v_0B_x-\left(\mathbf{v}\times\frac{\mathbf{E}}{c}\right)_z\right]\mathbf{e}_1\right. \\
&\quad \left.+\left[v_0B_y-\left(\mathbf{v}\times\frac{\mathbf{E}}{c}\right)_y\right]\mathbf{e}_2+\left[v_0B_z-\left(\mathbf{v}\times\frac{\mathbf{E}}{c}\right)_x\right]\mathbf{e}_3\right\} \\
&= \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\left[(\mathbf{v}\cdot\mathbf{B})\,\mathbf{e}_0+v_0\mathbf{B}-\left(\mathbf{v}\times\frac{\mathbf{E}}{c}\right)\right] \qquad v_0=c
\end{aligned}$$

19.2 Associative Product of Trivector T and Trivector T'

$$\begin{aligned}
\mathbf{TT}' &= (T^{012}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2+T^{031}\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1+T^{023}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3+T^{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) \\
&\quad \left(T'^{012}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2+T'^{031}\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1+T'^{023}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3+T'^{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\right) \\
&= T^{012}T'^{012}+T^{031}T'^{031}+T^{023}T'^{023}-T^{123}T'^{123} \\
&\quad +(T^{123}T'^{023}-T^{023}T'^{123})\mathbf{e}_0\mathbf{e}_1+\left(T^{123}T'^{031}-T^{031}T'^{123}\right)\mathbf{e}_0\mathbf{e}_2 \\
&\quad +\left(T^{123}T'^{012}-T^{012}T'^{123}\right)\mathbf{e}_0\mathbf{e}_3+\left(T^{023}T'^{031}-T^{031}T'^{023}\right)\mathbf{e}_1\mathbf{e}_2 \\
&\quad +\left(T^{012}T'^{023}-T^{023}T'^{012}\right)\mathbf{e}_3\mathbf{e}_2+\left(T^{012}T'^{031}-T^{031}T'^{012}\right)\mathbf{e}_2\mathbf{e}_3
\end{aligned}$$

$\mathbf{T}'\mathbf{T}$ = is the same as the above with signs reversed on bivectors. Therefore

$$\begin{aligned}
(\mathbf{TT}'-\mathbf{T}'\mathbf{T})/2 &= +(T^{123}T'^{023}-T^{023}T'^{123})\mathbf{e}_0\mathbf{e}_1+\left(T^{123}T'^{031}-T^{031}T'^{123}\right)\mathbf{e}_0\mathbf{e}_2 \\
&\quad +\left(T^{123}T'^{012}-T^{012}T'^{123}\right)\mathbf{e}_0\mathbf{e}_3+\left(T^{023}T'^{031}-T^{031}T'^{023}\right)\mathbf{e}_1\mathbf{e}_2 \\
&\quad +\left(T^{012}T'^{023}-T^{023}T'^{012}\right)\mathbf{e}_3\mathbf{e}_1+\left(T^{031}T'^{012}-T^{012}T'^{031}\right)\mathbf{e}_2\mathbf{e}_3
\end{aligned}$$

which is a bivector. Also

$$(\mathbf{TT}'+\mathbf{T}'\mathbf{T})/2=T^{012}T'^{012}+T^{031}T'^{031}+T^{023}T'^{023}-T^{123}T'^{123}$$

which is a scalar. The sum of the previous two calculations may be performed as follows:

Multiply trivectors by $-\mathbf{e}_5\mathbf{e}_5 = 1$ where $\mathbf{e}_5 = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$

$$\begin{aligned}
& \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\left(-\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\right)\left[T^{012}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + T^{031}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3 + T^{023}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + T^{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\right] \\
= \;& \mathbf{e}_5\left[T^{012}\mathbf{e}_3 + T^{031}\mathbf{e}_2 + T^{023}\mathbf{e}_1 + T^{123}\mathbf{e}_0\right] \\
= \;& \mathbf{e}_5\left[\mathbf{e}_0 u_0 + \mathbf{e}_1 u_1 + \mathbf{e}_2 u_2 + u_3\mathbf{e}_3\right] \qquad \mathbf{e}_0\mathbf{e}_0 = -1, \mathbf{e}_1\mathbf{e}_1 = 1, \text{etc.} \\
u_0 = \;& T^{123}, \quad u_1 = T^{023}, \quad u_2 = T^{031}, \quad u_3 = T^{012},
\end{aligned}$$

In general $\quad \mathbf{e}_5\,\mathrm{vec} = -\mathrm{vec}\,\mathbf{e}_5, \quad \mathbf{e}_5\mathrm{Biv} = \mathrm{Biv}\mathbf{e}_5, \quad \mathbf{e}_5\mathrm{Triv} = -\mathrm{Triv}\mathbf{e}_5,$

$$\mathbf{e}_5 QV = QV\mathbf{e}_5 \;\; (QV = S'\mathbf{e}_5)$$

QV abbreviation for quadvector.

Detail:

$$\mathbf{T}\mathbf{T}' = \mathbf{e}_5\left[u_0\mathbf{e}_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3\right]\mathbf{e}_5\left[u_0'\mathbf{e}_0 + u_1'\mathbf{e}_1 + u_2'\mathbf{e}_2 + u_3'\mathbf{e}_3\right] \tag{19.22}$$

$$\begin{aligned}
&= \left[u_0\mathbf{e}_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3\right]\left[u_0'\mathbf{e}_0 + u_1'\mathbf{e}_1 + u_2'\mathbf{e}_2 + u_3'\mathbf{e}_3\right] \\
\mathbf{T}\mathbf{T}' &= \underline{-u_0u_0'\mathbf{e}_0\mathbf{e}_0} + u_0u_1'\mathbf{e}_0\mathbf{e}_1 + u_0u_2'\mathbf{e}_0\mathbf{e}_2 + u_0u_3'\mathbf{e}_0\mathbf{e}_3 \\
&\quad + u_1u_0'\mathbf{e}_1\mathbf{e}_0 + \underline{u_1u_1'\mathbf{e}_1\mathbf{e}_1} + u_1u_2'\mathbf{e}_1\mathbf{e}_2 + u_1u_3'\mathbf{e}_1\mathbf{e}_3 \\
&\quad + u_2u_0'\mathbf{e}_2\mathbf{e}_0 + u_2u_1'\mathbf{e}_2\mathbf{e}_1 + \underline{u_2u_2'\mathbf{e}_2\mathbf{e}_2} + u_2u_3'\mathbf{e}_2\mathbf{e}_3 \\
&\quad + u_3u_0'\mathbf{e}_3\mathbf{e}_0 + u_3u_1'\mathbf{e}_3\mathbf{e}_1 + u_3u_2'\mathbf{e}_3\mathbf{e}_2 + \underline{u_3u_3'\mathbf{e}_3\mathbf{e}_3} \\
&= -\left(-u_0u_0' + u_1u_1' + u_2u_2' + u_3u_3'\right) \\
&\quad + \left(u_0u_1' - u_1u_0'\right)\mathbf{e}_0\mathbf{e}_1 + \left(u_0u_2' - u_2u_0'\right)\mathbf{e}_0\mathbf{e}_2 + \left(u_0u_3' - u_3u_0'\right)\mathbf{e}_0\mathbf{e}_3 \\
&\quad + \left(u_1u_2' - u_2u_1'\right)\mathbf{e}_1\mathbf{e}_2 + \left(u_3u_1' - u_1u_3'\right)\mathbf{e}_3\mathbf{e}_1 + \left(u_2u_3' - u_3u_2'\right)\mathbf{e}_2\mathbf{e}_3 \\
&= \text{scalar} + \text{bivector}
\end{aligned}$$

19.3 Parity

By the old hypothesis, it was assumed that all interactions are invariant under spatial inversion: that is under mirror reflection plus rotation by 180°. Lee and Yang concluded that no experiment had been performed that was sensitive to parity violating effects; They pointed out that in order to detect parity violation one would have to observe a pseudoscalar quantity.

Pseudoscalar: Invariant under rotation but changes sign under spatial inversion.

Evaluation of second Term II in Eq. (19.13), when space vectors $\mathbf{v}$ are replaced by space time vectors $\mathbf{V}$, that is,

$$\mathbf{v} = v_x\mathbf{e}_1 + v_y\mathbf{e}_2 + v_z\mathbf{e}_3 \quad \rightarrow \quad \mathbf{V} = c\mathbf{e}_0 + \mathbf{v}$$

Thus

$$\mathbf{F}_{ts}^{I} = q_s q_s k \mu_0 \left(\mathbf{V}_s \wedge \mathbf{r}\right) \bullet \mathbf{V}_t = q_t q_s k \mathbf{B}_s \bullet \mathbf{V}_t = q_t q_s k \left(\mathbf{B}_s \mathbf{V}_t - \mathbf{V}_t \mathbf{B}_s\right)/2 \tag{19.23}$$

$$\mathbf{F}_{ts}^{I} \overset{\text{yields}}{=} q_t q_s \mu_0 k \left[\left(\mathbf{v}_t \cdot \frac{\mathbf{E}_s}{c}\right)\mathbf{e}_0 + \mathbf{E}_s + \mathbf{v}_t \times \mathbf{B}_s\right] \tag{19.24}$$

The coefficient of $\mathbf{e}_0$ is the power required to push q_t with a velocity $\mathbf{v}_t$ into the field $\mathbf{E}_s$ or the work per unit time done by the field on $q_t\mathbf{v}_t$. The other two vector terms constitute the usual Lorentz force. The latter yields a positive sign for the force of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$ when both q_s and q_t are positive. When one is positive and the other negative, $\mathbf{F}_{ts}$ as given by Eqs. (19.10), (19.11) is negative, that is, attractive.

Now evaluate the third Term in Eq. (19.13), namely

$$\mathbf{F}_{ts}^{Ia} = q_t q_s \mu_0 k \left(\mathbf{V}_s \wedge \mathbf{r}\right) \wedge \mathbf{V}_t = q_t q_s k \mathbf{B}_s \wedge \mathbf{V}_t = q_t q_s k \left(\mathbf{B}_s \mathbf{V}_t + \mathbf{V}_t \mathbf{B}_s\right)/2 \tag{19.25}$$

We have replaced $\mathbf{V}_t \wedge \mathbf{B}_s$ by $\mathbf{B}_s \wedge \mathbf{V}_t$ to be consistent with our similar reversal to obtain the proper signs for the electromagnetic force, although
$\mathbf{V}_t \wedge \mathbf{B}_s \equiv \mathbf{B}_s \wedge \mathbf{V}_t$.

$$\mathbf{F}_{ts} = q_t q_s \mu_0 k \mathbf{e}_5 \left[\left(\mathbf{v}_t \cdot \mathbf{B}_s\right)\mathbf{e}_0 + c\mathbf{B}_s + \mathbf{v}_t \times \frac{\mathbf{E}_s}{c}\right] \tag{19.26}$$

The coefficient of $\mathbf{e}_0$

$$\mathbf{F}_{ts} = q_t q_s \mu_0 k \left(\mathbf{v}_t \cdot \mathbf{B}_s\right) = q_t q_s \mu_0 k \mathbf{V}_t \cdot \left(\mathbf{V}_s \times \mathbf{r}_{st}\right) \tag{19.27}$$

describes a new force. It is interpreted as the weak force.

When the Coulomb spatial spreading factor $k = 1/4\pi r^2$ is multiplied by the Yukawa factor $\mathbf{e}^{-\mu/r}$ associated with a massive field, the Yukawa factor $\mathbf{e}^{-\mu/r}$ appears naturally in the potential by solving Poisson's equation $\nabla^2\phi = -\mu/r$. So the Coulomb potential $1/4\pi r$ is replaced by $\mathbf{e}^{-\mu/r}/4\pi r$.

Eq. (19.26) becomes

$$\mathbf{F}_{ts} = q_t q_s \mu_0' k' \mathbf{e}_5 \left[\left(\mathbf{v}_t \cdot \mathbf{B}_s\right)\mathbf{e}_0 + c\mathbf{B}_s + \mathbf{v}_t \times \frac{\mathbf{E}_s}{c}\right] \tag{19.28}$$

We have replaced $k = 1/4\pi r^2$ by $k' = \mathbf{e}^{-\mu/r}/4\pi r^2$. One does not know in advance that massive particles (bosons) convey the interaction in Eq. (19.28), but experiment shows this to be so for the weak force. The force term is

$q_t q_s \mu_0' k' \gamma_t \gamma_s (\mathbf{v}_t \cdot \mathbf{B}) \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. When we express this as contribution to the potential it is written

$$\begin{aligned} \Phi &= -q_t q_s \mu_0 k' (\mathbf{v}_t \cdot \mathbf{B}_s) \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \qquad (19.29) \\ k' &= \mathrm{e}^{-\mu/r}/4\pi r \end{aligned}$$

We have replaced μ_0, Eq. (19.28), by μ_0' although experiment indicates that $\mu_0' = \mu_0$ within experimental error. In Eq. (19.29) $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ tells us that the term $(\mathbf{v}_t \cdot \mathbf{B}_s) \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ changes sign under inversion $(\mathbf{e}_1 \to -\mathbf{e}_1,\ \mathbf{e}_2 \to -\mathbf{e}_2,\ \mathbf{e}_3 \to -\mathbf{e}_3)$ without our examining
$\mathbf{v}_t \cdot \mathbf{B}_s$. $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is simply a label that tells what kind of a quantity $\mathbf{v}_t \cdot \mathbf{B}_s$ is. $\mathbf{v}_t \cdot \mathbf{B}_s$ at first glance looks like a scalar; however, since
$(\mathbf{v}_t \cdot \mathbf{B}_s\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)(\mathbf{v}_t \cdot \mathbf{B}_s\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) = -(\mathbf{v}_t \cdot \mathbf{B}_s)^2$ and $(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)^2 = -1$, it is not a scalar but a pseudo scalar, that is, its sign changes when the quantity is squared.

19.4 Remarks on Inversion

A 3-vector polar vector changes sign under inversion

$$\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$$

This may be seen by reversing the signs of the coefficients $x \to -x$,
$y \to -y,\ z \to -z$

$$\mathbf{r} \to \mathbf{r}' = -(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3)$$

or by reversing the sign of the unit vectors $\mathbf{e}_1 \to -\mathbf{e}_1,\ \mathbf{e}_2 \to -\mathbf{e}_2,\ \mathbf{e}_3 \to -\mathbf{e}_3$

The space time 4-vector behaves under inversion as:

$$\begin{aligned} \mathbf{s} &= s_0\mathbf{e}_0 + x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 \\ \text{as} \qquad \mathbf{s} \to \mathbf{s}' &= s_0\mathbf{e}_0 - (x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3) \end{aligned}$$

The space part changes sign. To change sign overall requires $s_0 \to -s_0$ or $\mathbf{e}_0 \to -\mathbf{e}_0$.

The electric field defined by

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^2}$$

changes sign under inversion.

The magnetic field defined by

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{\mathbf{v} \times \mathbf{r}}{r^3}$$

does not change sign under inversion, since under space inversion

$$\mathbf{v} = \dot{x}\mathbf{e}_1 + \dot{y}\mathbf{e}_2 + \dot{z}\mathbf{e}_3 \rightarrow \ -(\dot{x}\mathbf{e}_1 + \dot{y}\mathbf{e}_2 + \dot{z}\mathbf{e}_3)$$

changes sign since $\dot{x} = dx/dt$ changes sign if $x \rightarrow -x$ and $t \rightarrow t$.

Velocity changes sign if $x \rightarrow x$ and $t \rightarrow -t$. If both $x \rightarrow -x$ and $t \rightarrow -t$ then velocity does not change sign. Thus velocity does not change sign under space and time inversion.

The latter may be seen when the velocity is written as a 4-vector

$$V = \frac{(c\mathbf{e}_0 + \dot{x}\mathbf{e}_1 + \dot{y}\mathbf{e}_2 + \dot{z}\mathbf{e}_3)}{\sqrt{1 - v^2/c^2}} = \frac{(\mathbf{e}_0 c + \mathbf{v})}{\sqrt{1 - v^2/c^2}}$$

When written as a space-time vector, the behavior of velocity under space time inversion may be seen by examining the behavior of the unit vectors

$$\mathbf{e}_0 \rightarrow -\mathbf{e}_0, \qquad \mathbf{e}_1 \rightarrow -\mathbf{e}_1, \qquad \mathbf{e}_2 \rightarrow -\mathbf{e}_2, \qquad \mathbf{e}_3 \rightarrow -\mathbf{e}_3, \qquad e_0 e0 = -1$$

$$\begin{aligned}
\mathbf{FV} &= \left(\frac{E_x}{c}\mathbf{e}_0\mathbf{e}_1 + \frac{E_y}{c}\mathbf{e}_0\mathbf{e}_2 + \frac{E_z}{c}\mathbf{e}_0\mathbf{e}_3 + B_z\mathbf{e}_1\mathbf{e}_2 + B_y\mathbf{e}_3\mathbf{e}_1 + B_x\mathbf{e}_2\mathbf{e}_3\right) \\
&\quad (v_0\mathbf{e}_0 + v_x\mathbf{e}_1 + v_y\mathbf{e}_2 + v_z\mathbf{e}_3) \\
&= \frac{E_x}{c}v_0\mathbf{e}_1 + \frac{E_x}{c}v_x\mathbf{e}_0 + \frac{E_x}{c}v_y\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 - \frac{E_x}{c}v_z\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 \\
&\quad + \frac{E_y}{c}v_0\mathbf{e}_2 - \frac{E_y}{c}v_x\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + \frac{E_y}{c}v_y\mathbf{e}_0 + \frac{E_y}{c}v_z\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 \\
&\quad + \frac{E_z}{c}v_0\mathbf{e}_3 + \frac{E_z}{c}v_x\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 - \frac{E_z}{c}v_y\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + \frac{E_z}{c}v_z\mathbf{e}_0 \\
&\quad + B_z v_0\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 - B_z v_x\mathbf{e}_2 + B_z v_y\mathbf{e}_1 + B_z v_z\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\
&\quad + B_y v_0\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1 + B_y v_x\mathbf{e}_3 + B_y v_y\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 - B_y v_z\mathbf{e}_1 \\
&\quad + B_x v_0\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 + B_x v_x\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 - B_x v_y\mathbf{e}_3 + B_x v_z\mathbf{e}_2
\end{aligned}$$

$$\begin{aligned}
\mathbf{FV} &= v_0\left(\frac{E_x}{c}\mathbf{e}_1+\frac{E_y}{c}\mathbf{e}_2+\frac{E_z}{c}\mathbf{e}_3\right)+\left(\frac{E_x}{c}v_x+\frac{E_y}{c}v_y+\frac{E_z}{c}v_z\right)\mathbf{e}_0\\
&+\left(\frac{E_x}{c}v_y-\frac{E_y}{c}v_x\right)\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2+\left(\frac{E_z}{c}v_x-\frac{E_x}{c}v_z\right)\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1\\
&+\left(\frac{E_y}{c}v_z-\frac{E_z}{c}v_y\right)\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3\\
&+B_z v_0\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2+\left(B_x v_z-B_z v_x\right)\mathbf{e}_2+\left(B_z v_y-B_y v_z\right)\mathbf{e}_1\\
&+\left(B_z v_z+B_y v_y+B_x v_x\right)\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3+B_y v_0\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1+B_x v_0\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3\\
&+B_z v_0\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2+\left(B_y v_x-B_x v_y\right)\mathbf{e}_3\\
\mathbf{FV} &= v_0\frac{\mathbf{E}}{c}+\left(\mathbf{v}\cdot\frac{\mathbf{E}}{c}\right)\mathbf{e}_0+v_0\mathbf{e}_5\mathbf{B}-\mathbf{e}_5\left(\mathbf{v}\times\frac{\mathbf{E}}{c}\right)+\left(\mathbf{v}\times\mathbf{B}\right)+\mathbf{e}_5\left(\mathbf{v}\cdot\mathbf{B}\right)\mathbf{e}_0
\end{aligned}$$

For $v_0 = c$

$$\begin{aligned}
\mathbf{FV} &= \left(\mathbf{v}\cdot\frac{\mathbf{E}}{c}\right)\mathbf{e}_0+\mathbf{E}+\mathbf{v}\times\mathbf{B}+\mathbf{e}_5\left[\left(\mathbf{v}\cdot\mathbf{B}\right)\mathbf{e}_0+c\mathbf{B}-\left(\mathbf{v}\times\frac{\mathbf{E}}{c}\right)\right]\\
\mathbf{V} &= c\mathbf{e}_0+\mathbf{v}
\end{aligned}$$

$$\begin{aligned}
\mathbf{VF} &= \left(c\mathbf{e}_0+v_x\mathbf{e}_1+v_y\mathbf{e}_2+v_z\mathbf{e}_3\right)\left(\frac{E_x}{c}\mathbf{e}_0\mathbf{e}_1+\frac{E_y}{c}\mathbf{e}_0\mathbf{e}_2+\frac{E_z}{c}\mathbf{e}_0\mathbf{e}_3\right.\\
&\left.+B_z\mathbf{e}_1\mathbf{e}_2+B_y\mathbf{e}_3\mathbf{e}_1+B_x\mathbf{e}_2\mathbf{e}_3\right)\\
\mathbf{VF} &= -c\frac{E_x}{c}\mathbf{e}_1-c\frac{E_y}{c}\mathbf{e}_2-c\frac{E_z}{c}\mathbf{e}_3+cB_z\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2+cB_y\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1+cB_x\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3\\
&-v_x\frac{E_x}{c}\mathbf{e}_0-v_x\frac{E_y}{c}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2+v_x\frac{E_z}{c}\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1+v_xB_z\mathbf{e}_2-v_xB_y\mathbf{e}_3+v_xB_x\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\\
&+v_y\frac{E_x}{c}\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2-v_y\frac{E_y}{c}\mathbf{e}_0-v_y\frac{E_z}{c}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3-v_yB_z\mathbf{e}_1+v_yB_y\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3+v_yB_x\mathbf{e}_3\\
&-v_z\frac{E_x}{c}\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1+v_z\frac{E_y}{c}\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3-v_z\frac{E_z}{c}\mathbf{e}_0+v_zB_z\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3+v_zB_y\mathbf{e}_1-v_zB_x\mathbf{e}_2
\end{aligned}$$

$$\begin{aligned}
\mathbf{VF} &= -c\left(\frac{E_x}{c}\mathbf{e}_1+\frac{E_y}{c}\mathbf{e}_2+\frac{E_z}{c}\mathbf{e}_3\right)+\mathbf{e}_5 c\mathbf{B}-\left(\mathbf{v}\cdot\frac{\mathbf{E}}{c}\right)\mathbf{e}_0+\left(v_xE_z-v_zE_x\right)\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\\
&\quad+\left(v_xB_z-v_zB_x\right)\mathbf{e}_2+\left(v_yB_x-v_xB_y\right)\mathbf{e}_3+\left(v_zB_y-v_yB_x\right)\mathbf{e}_1\\
&\quad+\left(v_xB_x+v_yB_y+v_zB_z\right)\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3+\left(v_yE_x-v_xE_y\right)\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\\
&\quad+\left(v_z\frac{E_y}{c}-v_y\frac{E_z}{c}\right)\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3+\left(v_x\frac{E_z}{c}-v_z\frac{E_x}{c}\right)\mathbf{e}_0\mathbf{e}_3\mathbf{e}_1+\left(\mathbf{v}\cdot\mathbf{B}\right)\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\\
\mathbf{VF} &= -c\frac{\mathbf{E}}{c}-\left(\mathbf{v}\cdot\frac{\mathbf{E}}{c}\right)\mathbf{e}_0+c\mathbf{e}_5\mathbf{B}-\mathbf{e}_5\left(\mathbf{v}\times\frac{\mathbf{E}}{c}\right)-\left(\mathbf{v}\times\mathbf{B}\right)+\mathbf{e}_5\left(\mathbf{v}\cdot\mathbf{B}\mathbf{e}_0\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\mathbf{F}\bullet\mathbf{V}}{2} &= \frac{\mathbf{FV}-\mathbf{VF}}{2}=\mathbf{e}_5\left[\left(\mathbf{v}\cdot\frac{\mathbf{E}}{c}\right)\mathbf{e}_0+\mathbf{E}+\left(\mathbf{v}\times\mathbf{B}\right)\right]\\
\frac{\mathbf{F}\wedge\mathbf{V}}{2} &= \frac{\mathbf{FV}+\mathbf{VF}}{2}=\mathbf{e}_5\left[\left(\mathbf{v}\cdot\mathbf{B}\right)\mathbf{e}_0+c\mathbf{B}\right]\\
\mathbf{F}\bullet\mathbf{V} &= \frac{\mathbf{F}_s\mathbf{V}_t-\mathbf{V}_t\mathbf{F}_s}{2}=\mathbf{e}_5\left[\mathbf{e}_0+\mathbf{E}_s+\mathbf{v}_t\times\mathbf{B}_s\right]\\
\mathbf{F}\wedge\mathbf{V} &= \frac{\mathbf{F}_s\mathbf{V}_t+\mathbf{V}_t\mathbf{F}_s}{2}=\mathbf{e}_5\left[\left(\mathbf{v}_t\cdot\mathbf{B}_s\right)\mathbf{e}_0+c\mathbf{B}_s\right]
\end{aligned}$$

As we have seen, forces mediated by massless particles spead out in accordance with the geodesic spreading factor $1/r^2$. This in turn may be associated with a potential $1/r$ so $-\frac{d}{dr}\left(\frac{1}{r}\right)=\frac{1}{r^2}$.

Brief comment on massive particles as force mediators.

If the interacting particles are massive the potential must be modified by the Yukawa factor $\mathbf{e}^{-\mu/r}$ so that the force goes as $-\frac{d}{dr}\left(\frac{\mathbf{e}^{-\mu/r}}{r}\right)$. Assume energy loss proportional to mdr. Then

$$dm=-\mu m dr,\qquad \int\frac{dm}{m}=-\mu dr\qquad \log\frac{m}{m_0}=-\mu\left(r-r_0\right)\qquad \frac{m}{m_0}=\mathbf{e}^{-\mu(r-r_0)}$$

The potential change represents energy change. $m=m_0\mathbf{e}^{-\mu r}$ represents mass as a function of r, which in turn is energy change. Total energy at position r goes as $-\frac{1}{r}\mathbf{e}^{-\mu r}$, the product of the two factors. The subsequent force as a function of r goes as $-\frac{d}{dr}\left(\frac{\mathbf{e}^{-\mu r}}{r}\right)$.