15

ROTATION CURVE. MULTIVECTOR FORCES. TRIVECTOR, NEGATIVE GRAVITY FORCE. CONTRACTION

In the following, trivectors appear in parallel with vectors. In this work we are interested in vectors, but carry along the trivectors that appear in the formalism. Trivectors are involved in parity violation, (see Chapter 19). We have not investigated their possible role on a cosmic scale. See Chapter 19 for their appearance when discussing parity and the "electroweak force."

In this chapter, we show how negative gravity appears naturally as a force in nature and can account for the observed expansion of galaxies and the Universe as a whole.

15.1 Rotation Curve

We first show the origin of the idea of "dark matter and energy" by repeating the calculation of Pasachoff and Filippenko (Pasachoff and Filippenko 2006).

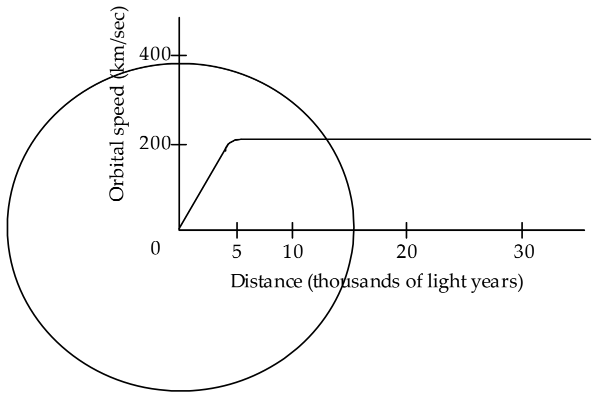


Fig. 15.1

Fig. 15.1 shows a view looking down on the plane of the Milky Way. The radial scale is thousands of light-years. An imaginary circle of radius of approximately 17×10^3 light-years is shown. The rotation curve is drawn along an arbitrary direction. An identical curve can be imagined for any radius with the sun, (not shown), located along one of them at a distance of 26,000 light-years from the center. A larger figure of the rotation curve itself is given in the book by Pasachoff and Filippenko.

According to the experimental rotation curve, the speed of mass in cm/sec at any distance from the center of the galaxy is

$$v = \left[(200 \text{ km/sec}) \left(10^5 \text{ cm/km}\right)\right] = 2 \times 10^7 \text{ cm/sec}$$

The mass is related to v by

$$M = v^2 R/G \qquad G = 6.673 \times 10^{-8} \text{ cm}^3/\text{g/sec}^2$$

At the position of the Sun, the distance from the center of the galaxy is:

$$\begin{aligned} R &= (26,000 \text{ lt-yr}) \left(9.5 \times 10^{17} \text{ cm/lt-yr}\right) \\ &= 2.47 \times 10^{22} \text{ cm} \end{aligned}$$

Therefore, out to the Sun

$$\begin{aligned} M &= \left(2 \times 10^7 \text{ cm/sec}\right)^2 \left(2.47 \times 10^{22} \text{ cm}\right) / \left(6.673 \times 10^{-8} \text{ cm}^3/\text{g/sec}^2\right) \\ &= \left(4 \times 10^{14} \times 2.47 \times 10^{22} \times 10^8 \text{ g}\right)/6.673 \\ &= 1.5 \times 10^{44} \text{ g} \end{aligned}$$

The mass of the sun is about 10^{33} grams, so

$$M = \frac{1.5 \times 10^{44} \text{ g}}{2 \times 10^{33} \text{ g/solar mass}} = 7.4 \times 10^{10} \text{ solar mass}$$

Thus, according to the experimental rotation curve, the mass of the galaxy is approximately 10^{11} solar masses.

To quote Pasachoff and Filippenko, (Pasachoff and Filippenko 2006, p. 379), "there are too few stars visible by a large margin" to account for this result. "Studies of the outer parts of the Milky Way Galaxy throughout the electromagnetic spectrum do not reveal sufficient quantities of material to account for the derived mass." Astronomers and physicists have introduced ad-hoc dark matter to account for the missing mass.

In the present chapter we dispense with dark matter and show that a real negative, dark trivector force is responsible for the galaxy and Universe expansion.

15.2 Multivector Interactions

The following discussion repeats some of the material in the text but allows a trivector, that occurs naturally, to be included in the formalism..

First, we write an equation that describes the force that a "source" multivector q_s whose velocity is $\mathbf{v}_s$ exerts on a "test" multivector q_t of like kind whose velocity is $\mathbf{v}_t$.[1] The roles of q_s and q_t are interchangeable. q_s, q_t are regarded as properties of space. The properties have a structure represented by any of the multivectors of space-time algebra: a scalar, a vector, a bivector,a trivector, a pseudo scalar or any combination thereof.

The force that $q_s\mathbf{v}_s$ exerts on $q_t\mathbf{v}_t$ consists of two parts: a direct force and an indirect force. In electromagnetism, what we call the direct force, is often called the mechanical force. It is the velocity dependent force that drives motors, generators, etc. The indirect force, also called the electromagnetic force, results from the time rate of change of the Poynting momentum field generated by the particles through their mutual interaction, as will be shown.

15.2.1 Direct Force

The direct force is a one-on-one interaction between q_s and q_t; no integrations over space are involved. It describes the force between particles when they reach their present or "instantaneous" position. When q_s and q_t are scalar electric charges, the direct force of q_s on q_t is given to a first approximation by the Lorentz equation

$$\mathbf{F}_{ts} = kq_tq_s\left(\mathbf{E}_s + \mathbf{v}_t \times \mathbf{B}'_s\right) \tag{15.1}$$

[1] By like kind we mean q_s and q_t are both scalars, both vectors, both bivectors etc. In this case the interaction always takes the form of a force. The formulas may be used to describe the interaction between multivectors of unlike kind. In this case the interaction can take the form of a torque, angular momentum, or a force. See Table A.

where the magnetic field $\mathbf{B}'_s$ is

$$\mathbf{B}'_s = \mu_0 \left(\mathbf{v}_s \times \mathbf{r}_{st}\right), \qquad \mathbf{E}_s = \mathbf{r}_{st}/\varepsilon_0, \qquad k = 1/4\pi r^3 \tag{15.2}$$

The distance between the "particles" is $\mathbf{r}_{st}$ measured from q_s to q_t.

Eq. (15.1) may be written as a single term if we replace $\mathbf{v}_s$ and $\mathbf{v}_t$ by space-time velocities $\mathbf{V}_s$ and $\mathbf{V}_t$. When calculating the direct force, the space-time vectors $\mathbf{V}_s$ and $\mathbf{V}_t$ are given by

$$\mathbf{V}_s = \left(c\mathbf{e}_0 + v_{sx}\mathbf{e}_1 + v_{sy}\mathbf{e}_2 + v_{sz}\mathbf{e}_3\right) = \left(c\mathbf{e}_0 + \mathbf{v}_s\right) \tag{15.3}$$

$$\mathbf{V}_t = \left(c\mathbf{e}_0 + v_{tx}\mathbf{e}_1 + v_{ty}\mathbf{e}_2 + v_{tz}\mathbf{e}_3\right) = \left(c\mathbf{e}_0 + \mathbf{v}_t\right) \tag{15.4}$$

$$v_s^2 = v_{sx}^2 + v_{sy}^2 + v_{sz}^2 \qquad v_t^2 = v_{tx}^2 + v_{ty}^2 + v_{tz}^2$$

Note that $\mathbf{V}_s$ and $\mathbf{V}_t$ may be converted to "4-velocities" by multiplying by γ_s and γ_t respectively where, as 4-velocities, $\gamma_s^2\mathbf{V}_s^2 = \gamma_t^2\mathbf{V}_t = -c^2$, an invariant.

$$\gamma_s = 1/\sqrt{1 - v_s^2/c^2} \qquad \gamma_t = 1/\sqrt{1 - v_t^2/c^2}$$

In the following, unless otherwise mentioned, $\mathbf{V}_s$ and $\mathbf{V}_t$ may be regarded as space-time velocities.

The total advance of the perihelion of Mercury is 5600 seconds of arc per century. All but 43 seconds of this advance is explained by the non-relativistic interaction of the other planets and sun with Mercury. Our purpose is to show that other sources taken together account for the missing 43 seconds of arc. It is only in one of the sources that the relativistic correction is important. It will be included when the source is evaluated.

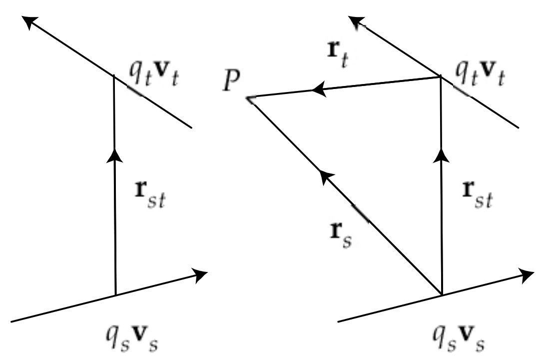


Fig. 15.2

The "direct" force of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$ when q_s and q_t are scalars is

$$\mathbf{F}_{ts}^{\text{dir}} = \mu_0 k q_s \mathbf{B}_s \bullet q_t \mathbf{V}_t = \mu_0 k q_s q_t \left(\mathbf{B}_s\mathbf{V}_t - \mathbf{V}_t\mathbf{B}_s\right)/2 \tag{15.5}$$

where $\mathbf{B}_s = (\mathbf{V}_s \wedge \mathbf{r}_{st}) = (\mathbf{V}_s \mathbf{r}_{st} + \mathbf{r}_{st} \mathbf{V}_s)/2$.

The direct force of $q_t \mathbf{v}_t$ on $q_s \mathbf{v}_s$ is obtained by interchanging s and t in Eq. (15.5) and replacing $\mathbf{r}_{st}$ by $\mathbf{r}_{ts}$. $\mathbf{r}_{ts} = -\mathbf{r}_{st}$.

$$\mathbf{F}_{st}^{\text{dir}} = \mu_0 k q_t q_s \mathbf{B}_t \bullet \mathbf{V}_s = \mu_0 k q_s q_t (\mathbf{B}_t \mathbf{V}_s - \mathbf{V}_s \mathbf{B}_t)/2 \tag{15.6}$$

where $\mathbf{B}_t = (\mathbf{V}_t \wedge \mathbf{r}_{ts})$.

Eqs. (15.5) and (15.6) do not satisfy Newton's third law, except for $\mathbf{v}_s$ parallel to $\mathbf{v}_t$, so in general $\mathbf{F}_{st} \neq \mathbf{F}_{ts}$. One of our initial goals is to obtain an accuracy of v^2/c^2 in force calculations since that degree of accuracy is required to calculate the observed advance in the perihelion of Mercury. However, equations will be set up so that a greater accuracy may be obtained by further expansion.

In Eq. (15.1) we substitute velocity dependent expressions for $\mathbf{E}_s$ and $\mathbf{E}_t$ which involve terms in v^2/c^2. This must be done for $\mathbf{E}_s$ and $\mathbf{E}_t$ since $\mathbf{B}_s$ and $\mathbf{B}_t$ are *ab initio* less than $\mathbf{E}_s$ and $\mathbf{E}_t$ by a factor v^2/c^2. The result will give both $\mathbf{E}$ and $\mathbf{B}$ to order v^2/c^2.

When q_t, q_s are arbitrary multivectors the direct force is still labeled $\mathbf{F}_{ts}^{\text{dir}}$, since it has the same form as the Faraday law, only the effective "charge," $(q_s q_t + q_t q_s)/2$, is different. The proportionality constants μ_0 and ε_0 will be different for each pair of like multivectors, but the relation $\mu_0 \varepsilon_0 = 1/c^2$ is assumed to hold. A superscript can be added to μ_0 and ε_0 to indicate which pair of multivectors are involved. The combination $(q_t q_s + q_s q_t)/2$ replaces $q_t q_s$ in Eq. (15.6). It is always a scalar for all multivectors of like kind, for example, both vectors, both bivectors and so on.

When q_s, q_t are not scalars an additional term

$$\mathbf{F}_{ts}^{\text{dir}}\,(\text{non-com}) = \mu_0 k (\overset{\text{Triv}}{\mathbf{B}_s \wedge \mathbf{V}_t}) (\overset{\text{Biv}}{q_s q_t - q_t q_s})/2 \tag{15.7}$$

also contributes to the net direct force of $q_s \mathbf{V}_s$ on $q_t \mathbf{V}_t$. The quantity $(q_s q_t - q_t q_s)/2$ is a bivector for all multivectors q_s, q_t of like kind. Since the force must be a vector, we introduce a procedure, to be described later, that will convert the product (triv)(biv) to a vector.

Table A. Multivector Structure of $(q_s q_t + q_t q_s)/2$, (Effective Charge) Designated by q_s/q_t. $q_s/q_t = (q_s q_t + q_t q_s)/2$

$q_s \backslash q_t$	S_t	V_t	B_t	T_t	Q_t
S_s	S_1	V_1	B_1	T_1	Q_1
V_s	V_1	S_2	T_2	B_2	0
B_s	B_1	T_2	S_3	T_3	B_3
T_s	T_1	B_2	T_3	S_4	0
Q_s	Q_1	0	B_3	0	S_5

Tabulation of the multivector structure of the quantity $(q_s q_t + q_t q_s)/2$ (coefficient of $(\mathbf{B}_s\mathbf{V}_t + \mathbf{V}_t\mathbf{B}_s)/2$, which is a trivector). q_s multivectors appear in the left hand column. q_t multivectors are listed in the top row. Example: If $q_t = V_s$ and $q_s = T_s$ then $(\mathbf{V}_s\mathbf{T}_t + \mathbf{T}_t\mathbf{V}_s)/2$ is a bivector B_2; Also if $q_t = B_t$ and $q_s = V_s$, then $(\mathbf{B}_s\mathbf{V}_t + \mathbf{V}_t\mathbf{B}_s)/2 = T_2$, a trivector as is shown in Table A. S_s is a scalar, for example electric charge.

V_s is a space-time vector. $P_s = m_s V_s$ is a space-time momentum . B_s is a bivector, for example the angular momentum $r \wedge P$ is also a bivector. T_s is a trivector. Q_s is a pseudo scalar, $\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 s_s$, also called a quadvector.

When q_s is a bivector and q_t a different kind of multivector, for example a trivector, the quantity $q_s q_t - q_t q_s$ is a bivector plus a scalar. See Table B and Table A for all possibilities.

Table B. Tabulation of the Multivector Structure of the Quantities $(q_s q_t - q_t q_s)/2$ (Effective "Charge") Insert masses, e.g. $q_s q_t \to m_s q_s m_t q_t$

$q_s \backslash q_t$	S_t	V_t	B_t	T_t	Q_t
S_s	0	0	0	0	0
V_s	0	B_1	V_1	$B{+}S$	T_1
B_s	0	$-V_1$	B_2	T_2	0
T_s	0	$-B-S$	$-T_2$	B_3	V_2
Q_s	0	$-T_1$	0	V_2	0

Table B shows the basic multivector structure of $(q_s q_t - q_t q_s)/2$ appearing in Term II, Eq.(15.13). For example, the entry V_s indicates that $(V_s B_t - B_t V_s)/2$ is a space-time vector, which is labeled V_1.

15.2.2 Indirect Force

The indirect force arises from the force that $q_s\mathbf{V}_s$ and $q_t\mathbf{V}_t$ exert on the multivector field, which is an electromagnetic field when two scalar electric charges are involved, and is expressed via the time rate of change of the Poynting momentum density vector generated jointly by $q_s\mathbf{V}_s$ and $q_t\mathbf{V}_t$, which momentum density must be integrated over all space and then differentiated with respect to time. When an equation is obtained for the force of the particles on the field, one must identify terms that effectively give the force of q_s on q_t and conversely the force of q_t on q_s. We will denote the indirect force that $q_s\mathbf{V}_s$ effectively exerts on $q_t\mathbf{V}_t$ by $\mathbf{F}_{ts}^{\text{ind}}$ and the indirect force of $q_t\mathbf{V}_t$ on $q_s\mathbf{V}_s$ by $\mathbf{F}_{st}^{\text{ind}}$. When both direct and indirect forces are calculated and added, we shall see that Newton's third law is satisfied up to terms in second order in $1/c$, specifically to v^2/c^2.

To repeat, each multivector force equation describes the interaction of multivector sources and consists of two parts. One part specifies the "direct" or point to point interaction. No integrations over space are involved. The other part which yields the indirect force is deduced from the total force in the field that is generated by the time rate of change of the total momentum in space. The momentum density is expressed by the mutual interaction terms in the Poynting momentum density vector which, after integration over all space and differentiation with respect to time, gives the total force in the field generated by the motion of the two "particles," the two multivectors.

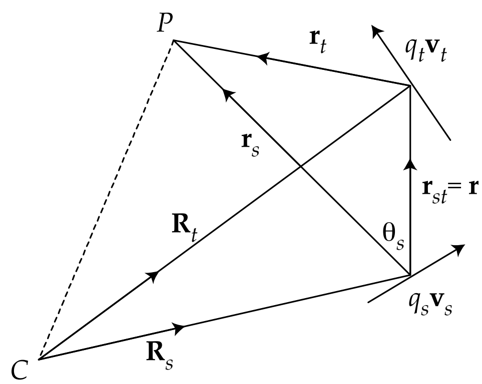
There is another radiation contribution to the force on each particle. It is the reactive force experienced by a particle on itself as it generates a time rate of change of field momentum. It is contained in the time rate of change of the Poynting momentum vector associated with a single particle.

The transport space-time geometry or algebra involves the velocity terms and distances, the algebra for which is represented by heavy case letters. The space-time algebra of the multivector being transported is represented by standard non-heavy case letters. The two algebras commute with each other but must finally be combined into a single algebra. The recipe for this, which we call "contraction" will be described later after the transport-interaction equation is written. The basic idea, however, is that contraction must be defined so that terms involving both algebras after contraction will represent a vector, that is, a force for all pairs of multivectors of like kind. The final result will be written in standard heavy case notation for vectors.

Also, in parallel with the vector force described above, there is a separate set of equations yielding a trivector. Each set of equations contains a direct and indirect component, both of which components are multiplied by an effective scalar charge $(q_sq_t + q_tq_s)/2$. This second group of equations also contains a direct and an indirect component each of which is multiplied by the bivector $(q_sq_t - q_tq_s)/2$. The "direct" force in each case is a point to point interaction manifested at the position of each particle. To obtain the indirect forces, on the other hand, the time rate of change of momentum imparted to the field must be integrated over all space and the effect of one particle (multivector) on the other identified after completion of the integration. This integration and the force identification resulting therefrom will be described in Part II.

The vector force may be mediated by massless or massive particles. The electromagnetic force is conveyed by massless particles, namely photons. The strong force is conveyed by massive particles.

Electric charges do not exist in isolation but are associated with a mass, or have a property called mass associated with the field. When we speak of charges, an associated mass is implied but not always explicitly displayed.



$CP = \mathrm{R}_s + \mathrm{r}_s = \mathrm{R}_t + \mathrm{r}_t$
$\mathrm{r}_s = \mathrm{r}_{st} + \mathrm{r}_t$
$\mathrm{r}_t = \mathrm{r}_s - \mathrm{r}_{st}$

Fig. 15.3

In addition, multivectors deliver a direct and indirect angular momentum to the field which may be taken about an arbitrary point, C. The indirect mutual angular momentum of two multivectors must be integrated over all space. By taking the time derivative of this result we obtain the total torque delivered to the field, from which, by dividing by the lever arms to the arbitrary point C we obtain an additional reactive force on each multivector. These statements also apply to non-commutative interaction to be described later. They also apply to interactions between different kinds of multivectors, for example the interaction of a vector and a trivector or a bivector. These remarks will be clear in the details that follow.

15.3 General Formulation of the Forces Generated by $q_s\mathbf{v}_s$ and $q_t\mathbf{v}_t$.

We begin with the law of direct interaction as given by Eq. (15.8). $\mathbf{F}_{ts}$ means force of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$. The following equations apply to both the direct and the indirect Poynting force of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$. When describing the direct force, the velocity symbols are given by Eq. (15.15) below. The form of the field input is given by $\mathbf{B}_s$ shown in Eq. (15.8) below.

Now adopt a generalized force law as the product of $\mathbf{B}_s$ and $\mathbf{V}_s$, not simply the inner or dot product that appears in the standard Biot-Savart Law. This choice is consistent with the algebra. See the introduction for detailed explanation.

In the following, we list the vector force term and the trivector terms in each equation.

$$\mathbf{F}_{ts} = \mu_0 k q_s \mathbf{B}_s q_t \mathbf{V}_t \qquad \mathbf{B}_s = \mu_0 k \mathbf{V}_s \wedge \mathbf{r}_{st}, \qquad k = 1/4\pi r^3 \tag{15.8}$$

Write Eq. (15.8) as follows. q_s, q_t belong to the property algebra. $\mathbf{B}_s$ and $\mathbf{V}_t$ belong to the carrier algebra.

$$\mathbf{F}_{ts} = \mu_0 k \left[\frac{q_s \overset{1}{\mathbf{B}_s} q_t \mathbf{V}_t - q_t \overset{3}{\mathbf{V}_t} q_s \mathbf{B}_s}{2} + \frac{q_s \overset{4}{\mathbf{B}_s} q_t \mathbf{V}_t + q_t \overset{2}{\mathbf{V}_t} q_s \mathbf{B}_s}{2} \right] \tag{15.9}$$

Eq. (15.9) leads to Eq. (15.13). Note that Eq. (15.8) is an extension or generalization of Faraday's Law in that it allows for both the dot product (inner product) and the wedge product (outer product) between $\mathbf{B}$ and $\mathbf{V}$. The latter option introduces trivector terms.

We may proceed from Eq. (15.9) to Eq. (15.10) by expressing each product in Eq. (15.9) in terms of its dot and wedge products.

$$\begin{aligned}
\mathbf{F}_{ts} \;=\; & \frac{1}{2}\Big[(\mathbf{B}_s\mathbf{V}_t \overset{1a}{-} \mathbf{V}_t\mathbf{B}_s)q_s q_t + (\mathbf{B}_s\mathbf{V}_t \overset{1b}{+} \mathbf{V}_t\mathbf{B}_s)q_s q_t \\
& + (\mathbf{V}_t\mathbf{B}_s \overset{2a}{-} \mathbf{B}_s\mathbf{V}_t)q_t q_s + (\mathbf{V}_t\mathbf{B}_s \overset{2b}{+} \mathbf{B}_s\mathbf{V}_t)q_t q_s \\
& - (\mathbf{V}_t\mathbf{B}_s \overset{3a}{-} \mathbf{B}_s\mathbf{V}_t)q_t q_s - (\mathbf{V}_t\mathbf{B}_s \overset{3b}{+} \mathbf{B}_s\mathbf{V}_t)q_t q_s \\
& + (\mathbf{B}_s\mathbf{V}_t \overset{4a}{-} \mathbf{V}_t\mathbf{B}_s)q_s q_t + (\mathbf{B}_s\mathbf{V}_t \overset{4b}{+} \mathbf{V}_t\mathbf{B}_s)q_s q_t \Big]
\end{aligned} \tag{15.10}$$

That is, in the dot (inner) and wedge (outer) notation,

$$\begin{aligned}\mathbf{F}_{ts} &= \Big[(\mathbf{B}_s \overset{1a}{\bullet} \mathbf{V}_t)q_s q_t + (\mathbf{B}_s \overset{1b}{\wedge} \mathbf{V}_t)q_s q_t \\ &\quad -(\mathbf{V}_t \overset{2a}{\bullet} \mathbf{B}_s)q_t q_s + (\mathbf{V}_t \overset{2b}{\wedge} \mathbf{B}_s)q_t q_s \\ &\quad -(\mathbf{V}_t \overset{3a}{\bullet} \mathbf{B}_s)q_t q_s - (\mathbf{V}_t \overset{3b}{\wedge} \mathbf{B}_s)q_t q_s \\ &\quad +(\mathbf{B}_s \overset{4a}{\bullet} \mathbf{V}_t)q_s q_t + (\mathbf{B}_s \overset{4b}{\wedge} \mathbf{V}_t)q_s q_t\Big] \end{aligned} \tag{15.11}$$

Now use

$$\mathbf{B}_s \bullet \mathbf{V}_t = -\mathbf{V}_t \bullet \mathbf{B}_s, \qquad \mathbf{B}_s \wedge \mathbf{V}_t = \mathbf{V}_t \wedge \mathbf{B}_s$$

then

$$\begin{aligned}\mathbf{F}_{ts} &= (\mathbf{B}_s \bullet \overset{1}{\mathbf{V}}_t)\, q_s q_t + (\mathbf{B}_s \wedge \overset{2}{\mathbf{V}}_t)\, q_s q_t \\ &\quad -(\mathbf{B}_s \bullet \overset{4}{\mathbf{V}}_t)\, q_t q_s + (\mathbf{B}_s \wedge \overset{3}{\mathbf{V}}_t)\, q_t q_s \\ &\quad + (\mathbf{B}_s \bullet \overset{1}{\mathbf{V}}_t)\, q_t q_s - (\mathbf{B}_s \wedge \overset{2}{\mathbf{V}}_t)\, q_t q_s \\ &\quad + (\mathbf{B}_s \bullet \overset{4}{\mathbf{V}}_t)\, q_s q_t - (\mathbf{B}_s \wedge \overset{3}{\mathbf{V}}_t)\, q_s q_t \end{aligned} \tag{15.12}$$

to obtain

$$\begin{aligned}\mathbf{F}_{ts}^{\mathrm{dir}} &= \mu_0 k \Big[\overset{\text{I}}{\overset{\text{Vec}}{(\mathbf{B}_s \bullet \mathbf{V}_t)}\overset{\text{Scalar}}{(q_s q_t + q_t q_s)/2}} + \overset{\text{II}}{\overset{\text{Triv}}{(\mathbf{B}_s \wedge \mathbf{V}_t)}\overset{\text{Biv}}{(q_s q_t - q_t q_s)/2}} \\ &\quad + \overset{\text{III}}{\overset{\text{Triv}}{(\mathbf{B}_s \wedge \mathbf{V}_t)}\overset{\text{Scalar}}{(q_s q_t + q_t q_s)/2}} + \overset{\text{IV}}{\overset{\text{Vec}}{(\mathbf{B}_s \bullet \mathbf{V}_t)}\overset{\text{Biv}}{(q_s q_t - q_t q_s)/2}}\Big] \end{aligned} \tag{15.13}$$

To obtain Eq. (15.13), one may dispense with the above procedure and simply note that if $\mathbf{B}_s \bullet \mathbf{V}_t \equiv (\mathbf{B}_s\mathbf{V}_t - \mathbf{V}_t\mathbf{B}_s)/2$, a vector, is to form a vector force between two like property multivectors q_t and q_s, then the property multivectors must form a scalar. The combination of $(q_t q_s + q_s q_t)/2$ is a scalar for any pair of like multivectors and therefore must multiply the vector Term I shown in Eq. (15.13). Thus we can regard Eq. (15.13) as our starting point and by-pass Eqs. (15.10) and (15.11) leading up to it.

Thus, Term I does not require contraction since it is already a vector.

Term II requires contraction to form a vector. Term I plus Term II after contraction gives the full vector force.

Referring to the second term in Eq. (15.13), the combination $(\mathbf{B}_s\mathbf{V}_t + \mathbf{V}_t\mathbf{B}_s)/2$ is a trivector. The charge coefficient $(q_tq_s - q_sq_t)/2$ is always a bivector for any pair of like multivectors q_s, q_t. The two algebras involved as products in Term II must be converted to a single algebra in such a way that the result is a vector. We define such a process shortly and call it contraction. "Contraction" converts Term II to a vector (see item, Table IV). Term I + Term II describes the full vector force.

Term III in Eq. (15.13) is a trivector. Contraction of Term III is not necessary since the property multiplier is a scalar.

Contraction of Term IV yields a trivector (item 3, Table III). Thus Term III plus Term IV give the full trivector.

15.3.1 Contraction of Terms

In summary, after evaluation terms II and IV must be contracted.

Likewise the direct vector force of $q_t\mathbf{V}_t$ on $q_s\mathbf{V}_s$ is, interchanging subscripts s and t in Eq. (15.13),

$$\begin{aligned}\mathbf{F}_{st}^{\text{dir}} &= \mu_0 k\Bigg[\overset{\text{Vec}}{(\mathbf{B}_t \bullet \mathbf{V}_s)}\overset{\text{Scalar}}{\left(\overset{1a}{q_tq_s} + \overset{3a}{q_sq_t}\right)}/2 + \overset{\text{Triv}}{(\mathbf{B}_t \wedge \mathbf{V}_s)}\overset{\text{Biv}}{\left(\overset{1b}{q_tq_s} - \overset{3b}{q_sq_t}\right)}/2 \\ &\quad + \overset{\text{Triv}}{(\mathbf{B}_t \wedge \mathbf{V}_s)}\overset{\text{Scalar}}{\left(\overset{4b}{q_tq_s} + \overset{2b}{q_sq_t}\right)}/2 + \overset{\text{Vec}}{(\mathbf{B}_t \bullet \mathbf{V}_s)}\overset{\text{Biv}}{\left(\overset{4a}{q_tq_s} - \overset{2a}{q_sq_t}\right)}/2\Bigg] \end{aligned} \tag{15.14}$$

the terms of which must also be contracted.

To now obtain the direct force of $q_s\mathbf{V}_s$ on $q_t\mathbf{V}_t$ and vice versa substitute the following in Eqs. (15.13) and (15.14).

$$\begin{aligned}\mathbf{V}_t &= (c\mathbf{e}_0 + v_{tx}\mathbf{e}_1 + v_{ty}\mathbf{e}_2 + v_{tz}\mathbf{e}_3) = (c\mathbf{e}_0 + \mathbf{v}_t) \\ \mathbf{V}_s &= (c\mathbf{e}_0 + v_{sx}\mathbf{e}_1 + v_{sy}\mathbf{e}_2 + v_{sz}\mathbf{e}_3) = (c\mathbf{e}_0 + \mathbf{v}_s)\end{aligned} \tag{15.15}$$

and

$$\begin{aligned}\mathbf{B}_s &= \mu_0 k\,(\mathbf{V}_s \wedge \mathbf{r}_{st}) \\ \mathbf{B}_t &= \mu_0 k\,(\mathbf{V}_t \wedge \mathbf{r}_{ts}) \qquad k = 1/4\pi r^3\end{aligned} \tag{15.16}$$

The following shows details for evaluation of the bivector Terms in Eq. (15.13) and (15.14).

$$
\begin{aligned}
& (q_s q_t - q_t q_s)/2 = \\
& M_s m_t \left(ce_0 + v'_{sx}e_1 + v'_{sy}e_2 + v'_{sz}e_3\right)\left(ce_0 + v'_{tx}e_1 + v'_{ty}e_2 + v'_{tz}e_3\right) \\
= \; & M_s m_t \left[\left(cv'_{tx} - cv'_{sx}\right) e_0 e_1 + \left(cv'_{ty} - cv'_{sy}\right) e_0 e_1 + \left(cv'_{tz} - cv'_{sz}\right) e_0 e_3 \right. \\
& \left. + \left(v'_{sx}v'_{ty} - v'_{tx}v'_{sy}\right) e_1 e_2 + \left(v'_{sz}v'_{tx} - v'_{sx}v'_{tz}\right) e_3 e_1 + \left(v'_{sy}v'_{tz} - v'_{sz}v'_{ty} e_2 e_3\right)\right] \\
& -
\end{aligned}
$$

$$
\begin{aligned}
& \left(q_t q_s - q_s q_t\right)/2 = \\
& M_s m_t \left[e_0 e_1 c \left(v'_{sx} - v'_{tx}\right) + e_0 e_2 c \left(v'_{sy} - v'_{ty}\right) + e_0 e_3 c \left(v'_{sz} - v'_{tz}\right) \right. \\
& \left. + e_1 e_2 \left(\mathbf{v}_t \times \mathbf{v}_s\right)_z + e_3 e_1 \left(\mathbf{v}_t \times \mathbf{v}_s\right)_y + e_2 e_3 \left(\mathbf{v}_t \times \mathbf{v}_s\right)_x\right]
\end{aligned}
$$

$\gamma'_s \gamma'_t$ may be introduced into the above to make the velocity terms relativistic, $\gamma'_s \gamma'_t$ may be different from $\gamma_s \gamma_t$. Primes are added temporarily, for bookkeeping purposes, to distinguish the property velocities from the carrier velocities. Later, the primes on the velocity terms will be removed and all velocity terms will be treated as equal.

Evaluation of $\mathbf{B}_s \bullet \mathbf{V}_t$ and $\mathbf{B}_s \wedge \mathbf{V}_t$ gives

$$
\begin{aligned}
\mathbf{B}_s \bullet \mathbf{V}_t &= k\mu_0 \left[\mathbf{e}_0 c \left(\mathbf{v}_t \cdot \mathbf{r}_{st}\right) + c^2 \mathbf{r}_{st} + \mathbf{v}_t \times \left(\mathbf{v}_s \times \mathbf{r}_{st}\right)\right] && (15.17) \\
\mathbf{B} \wedge \mathbf{V}_t &= k\mu_0 \mathbf{e}_5 \left\{\mathbf{e}_0 \left[\mathbf{v}_t \cdot \left(\mathbf{v}_s \times \mathbf{r}_{st}\right)\right] + c\left[\left(\mathbf{v}_s \times \mathbf{r}_{st}\right) - \left(\mathbf{v}_t \times \mathbf{r}_{st}\right)\right]\right\} && (15.18) \\
k &= 1/4\pi r^3.
\end{aligned}
$$

As previously stated, the "transport" algebra is written in heavy case letters, as in Eqs. (15.17)and (15.18), and the "message" or "property" entities q_s, q_t, by non-heavy case standard print.

15.3.2 Vector Force

From Eqs. (15.13), (15.14) the total direct vector force of $q_s \mathbf{V}_s$ on $q_t \mathbf{V}_t$, that is, omitting the trivector force, for multivectors of like kind is the sum

$$
\mathbf{F}^{\text{dir}}_{ts}\,(\text{vec}) = \mu_0 k \left[(\mathbf{B}_s \overset{\text{Vec}}{\bullet} \mathbf{V}_t)(q_s q_t \overset{\text{Scalar}}{+} q_t q_s)/2 + (\mathbf{B}_s \overset{\text{Triv}}{\wedge} \mathbf{V}_t)(q_s q_t \overset{\text{Biv}}{-} q_t q_s)/2\right] \tag{15.19}
$$

The direct force of $q_t \mathbf{V}_t$ on $q_s \mathbf{V}_s$ is

$$
\mathbf{F}^{\text{dir}}_{st}\,(\text{vec}) = \mu_0 k \left[(\mathbf{B}_s \overset{\text{Vec}}{\bullet} \mathbf{V}_t)(q_t q_s \overset{\text{Scalar}}{+} q_s q_t)/2 + (\mathbf{B}_s \overset{\text{Triv}}{\wedge} \mathbf{V}_t)(q_t q_s \overset{\text{Biv}}{-} q_s q_t)/2\right] \tag{15.20}
$$

where

$$\mathbf{B}_s \bullet \mathbf{V}_t = (\mathbf{B}_s\mathbf{V}_t - \mathbf{V}_t\mathbf{B}_s)/2, \qquad \mathbf{B}_s \wedge \mathbf{V}_t = (\mathbf{B}_s\mathbf{V}_t + \mathbf{V}_t\mathbf{B}_s)/2 \tag{15.21}$$

As stated, the second term in Eqs. (15.19) and (15.20) must be "contracted" to yield a vector. Since the combination $(q_s q_t + q_t q_s)/2$ is always a scalar the first term does not require contraction.

The total direct trivector effect of $q_s\mathbf{V}_s$ on $q_t\mathbf{V}_t$ for multivectors q_s, q_t of like kind is given by, (second line in Eq. (15.13)),

$$\mathbf{F}_{ts}^{\text{dir}}\,(\text{triv}) = \mu_0 k \left[(\mathbf{B}_s \overset{\text{Triv}}{\wedge} \mathbf{V}_t)(\overset{\text{Scalar}}{q_s q_t + q_t q_s})/2 + (\mathbf{B}_s \overset{\text{Vec}}{\bullet} \mathbf{V}_t)(\overset{\text{Biv}}{q_s q_t - q_t q_s})/2 \right] \tag{15.22}$$

To obtain the corresponding direct trivector force of $q_t\mathbf{V}_t$ on $q_s\mathbf{V}_s$, interchange s and t in the above, to obtain, (second line in Eq. (15.14)),

$$\mathbf{F}_{st}^{\text{dir}}\,(\text{triv}) = \mu_0 k \left[(\mathbf{B}_t \overset{\text{Triv}}{\wedge} \mathbf{V}_s)(\overset{\text{Scalar}}{q_t q_s + q_s q_t})/2 + (\mathbf{B}_t \overset{\text{Vec}}{\bullet} \mathbf{V}_s)(\overset{\text{Biv}}{q_t q_s - q_s q_t})/2 \right] \tag{15.23}$$

The above quantities from the original Eqs. (15.13) and (15.14) have been separated in this way for clarity and emphasis.

For gravity, by hypothesis, the 4-momenta $P_s$and P_t are "gravitational charges:" Thus

$$\begin{aligned}
(q_s q_t + q_t q_s)/2 &= (P_s P_t + P_t P_s)/2 \\
&= M_s m_t \left[-c^2 + (v_s' \cdot v_t')\right] \\
&= \left[-M_s m_t c^2 + p_s' \cdot p_t'\right] \\
&= -c^2 \left[M_s m_t + \frac{v_s' \cdot v_t'}{c^2}\right]
\end{aligned}$$

We repeat, the “transport” algebra is written in heavy case letters and the “message” or “property” entities q_s, q_t, by non-heavy case standard print.

Primes are placed on the velocities occurring in P_s and P_t but will later be the same as the carrier velocities $\mathbf{v}_s$ and $\mathbf{v}_t$.

$$\begin{aligned}
P_s &= M_s(e_0 c + v_s') & P_t &= m_t(e_0 c + v_t') \\
P_s &= e_0 M_s c + p_s' & P_t &= e_0 m_t c + p_t' \\
p_s' &= M_s v_s' & p_t' &= m_t v_t'
\end{aligned}$$

$$
\begin{aligned}
& q_s q_t - q_t q_s)/2 = P_s P_t - P_t P_s/2 = \\
& M_s m_t \left(ce_0 + v'_{sx}e_1 + v'_{sy}e_2 + v'_{sz}e_3\right)\left(ce_0 + v'_{tx}e_1 + v'_{ty}e_2 + v'_{tz}e_3\right) \\
= \; & M_s m_t \left[\left(cv'_{tx} - cv'_{sx}\right) e_0 e_1 + \left(cv'_{ty} - cv'_{sy}\right) e_0 e_1 + \left(cv'_{tz} - cv'_{sz}\right) e_0 e_3 \right. \\
& \left. + \left(v'_{sx}v'_{ty} - v'_{tx}v'_{sy}\right) e_1 e_2 + \left(v'_{sz}v'_{tx} - v'_{sx}v'_{tz}\right) e_3 e_1 + \left(v'_{sy}v'_{tz} - v'_{sz}v'_{ty} e_2 e_3\right)\right]
\end{aligned}
$$

For further details see Section 8.9

15.4 Contraction (a repeat)

The products in the preceding equations for direct and indirect forces must be combined into a common algebra. We now define a procedure that always makes the second term in Eqs. (15.19) and (15.20) a vector and the second term in Eqs. (15.22) and (15.23) a trivector for every pair of multivectors q_s, q_t of like kind, that is, both scalars, both vectors, etc. $\mathbf{B}_s \wedge \mathbf{V}_t$ is a trivector and $\mathbf{B}_s \bullet \mathbf{V}_t$ is a vector. $(q_s q_t + q_t q_s)/2$ is always a scalar and $(q_s q_t - q_t q_s)/2$ is always a bivector, except when q_s, q_t are scalars, in which case it is zero.

Given two multivectors, $\mathbf{A}$, a trivector and B, a bivector, each from different commuting algebras, the combination $\left(\mathbf{A}^{-1}B^{-1} + B^{-1}\mathbf{A}^{-1}\right)/2$ will yield a vector if we reciprocate all of the unit vectors in $\mathbf{A}$ and simultaneously reverse their order and do the same for the unit vectors in B. Then change the unit vectors in B to match those in $\mathbf{A}$. That this process must be carried for the multivectors $\mathbf{A}$ and $\mathbf{B}$ is denoted by the negative exponent -1. For example, in $\mathbf{A}^{-1}$,

$$
\mathbf{e}_1 \;\rightarrow\; \mathbf{e}_1^{-1} = \mathbf{e}_1, \qquad \mathbf{e}_0 \rightarrow \mathbf{e}_0^{-1} = -\mathbf{e}_0; \qquad \mathbf{e}_1\mathbf{e}_2 \rightarrow \mathbf{e}_2^{-1}\mathbf{e}_1^{-1} = \mathbf{e}_2\mathbf{e}_1;
$$
$$
\mathbf{e}_0\mathbf{e}_1 \rightarrow \mathbf{e}_1^{-1}\mathbf{e}_0^{-1} \rightarrow (\mathbf{e}_1)(-\mathbf{e}_0) = -\mathbf{e}_1\mathbf{e}_0
$$

Now do the same for the non-heavy case unit vectors in B. After completion express the result of B^{-1} in heavy case letters and combine them with those of $\mathbf{A}^{-1}$ as members of the heavy case elements of $\mathbf{A}$. We adopt this contraction recipe for all pairs of multivectors $\mathbf{A}$ and B. The notation $\mathbf{F}_{ts}^{\text{dir}}$ (non-com) pertains to the force associated with $(q_s q_t - q_t q_s)/2$, that is, when q_s and q_t do not commute.

15.4.1 Indirect Force

We now write the equations that give the force that q_t and q_s jointly exert on the field. Evaluation of these forces will contain the time rate of change of the momentum fed into the field through the joint contribution of $q_t\mathbf{V}_t$ and $q_s\mathbf{V}_s$ and described by the time rate of change of the Poynting momentum vector. q_t and q_s will in turn experience a reactive force. The negative of the sum of the two reactive forces will equal the net force that the two charges exert on the field. From our expression for the total force that $q_t\mathbf{V}_t$ and $q_s\mathbf{V}_s$ exert on the field, we will identify terms that give the reactive force on q_t and q_s. The negative of the reactive force on q_t will be identified as a force due to the motion of q_s. We call this the indirect force of q_s on q_t and will add it to the direct force of q_s on q_t to obtain the total force that q_s exerts on q_t. Likewise, the negative of terms identified with the reactive force on q_s will represent an indirect force that $q_t\mathbf{V}_t$ exerts on $q_s\mathbf{V}_s$.

To obtain the indirect force density, that is, the electromagnetic force density, delivered to the field we use the general structures expressed by Eqs. (15.19), (15.20); (15.22), and (15.23), but with $\mathbf{B}_s$ and $\mathbf{B}_t$ replaced by $\mathbf{F}_s$ and $\mathbf{F}_t$, [Eqs. (15.32) and (15.33)], and $\mathbf{V}_t$, $\mathbf{V}_s$ replaced by $\mathbf{V}''_t$ and $\mathbf{V}''_s$, as defined by Eqs. (15.26, 15.27), so that

$$\begin{aligned}\mathbf{F}_{ts}^{\text{indir, field}} = \mu_0 k \Bigg[& \overset{\text{I}}{\overset{\text{Vec}}{(\mathbf{F}_s \bullet \mathbf{V}''_t)}\overset{\text{Scalar}}{(q_s q_t + q_t q_s)}/2} + \overset{\text{II}}{\overset{\text{Vec}}{(\mathbf{F}_s \wedge \mathbf{V}''_t)}\overset{\text{Biv}}{(q_s q_t - q_t q_s)}/2} \\ & + \overset{\text{III}}{\overset{\text{Triv}}{(\mathbf{F}_s \wedge \mathbf{V}''_t)}\overset{\text{Scalar}}{(q_s q_t + q_t q_s)}/2} + \overset{\text{IV}}{\overset{\text{Vec}}{(\mathbf{F}_s \bullet \mathbf{V}''_t)}\overset{\text{Biv}}{(q_s q_t - q_t q_s)}/2} \Bigg] \end{aligned} \tag{15.24}$$

and

$$\begin{aligned}\mathbf{F}_{st}^{\text{indir, field}} = \mu_0 k \Big[& \overset{\text{Vec}}{(\mathbf{F}_t \bullet \mathbf{V}''_s)}\overset{\text{Scalar}}{(q_t q_s + q_s q_t)}/2 + \overset{\text{Triv}}{(\mathbf{F}_t \wedge \mathbf{V}''_s)}\overset{\text{Biv}}{(q_t q_s - q_s q_t)}/2 \\ & + \overset{\text{Triv}}{(\mathbf{F}_t \wedge \mathbf{V}''_s)}\overset{\text{Scalar}}{(q_t q_s + q_s q_t)}/2 + \overset{\text{Vec}}{(\mathbf{F}_t \bullet \mathbf{V}''_s)}\overset{\text{Biv}}{(q_t q_s - q_s q_t)}/2 \Big] \end{aligned} \tag{15.25}$$

$$\mathbf{V}''_s = q_s\left(\mathbf{V}_s + \mathbf{e}_5\mathbf{U}_s\right) \tag{15.26}$$

$$\mathbf{V}''_t = q_t\left(\mathbf{V}_t + \mathbf{e}_5\mathbf{U}_t\right) \tag{15.27}$$

$$\mathbf{V}_s = \left(\mathbf{e}_0 \mathrm{div}\frac{\mathbf{E}_s}{c} - \frac{1}{c^2}\frac{\partial \mathbf{E}_s}{\partial t} + \mathrm{curl}\mathbf{B}_s\right) \tag{15.28}$$

$$\mathbf{U}_s = \left(-\mathbf{e}_0 \mathrm{div}\mathbf{B}_s + \frac{1}{c}\frac{\partial \mathbf{B}_s}{\partial t} + \mathrm{curl}\frac{\mathbf{E}_s}{c}\right) \tag{15.29}$$

$$\mathbf{V}_t = \left(\mathbf{e}_0 \mathrm{div}\frac{\mathbf{E}_t}{c} - \frac{1}{c^2}\frac{\partial \mathbf{E}_t}{\partial t} + \mathrm{curl}\mathbf{B}_t\right) \tag{15.30}$$

$$\mathbf{U}_t = \left(-\mathbf{e}_0 \mathrm{div}\mathbf{B}_t + \frac{1}{c}\frac{\partial \mathbf{B}_t}{\partial t} + \mathrm{curl}\mathbf{E}_t\right) \tag{15.31}$$

The sum of Eqs. (15.24) and (15.25) will generate the Poynting vector, $(\mathbf{E}_s + \mathbf{E}_t) \times (\mathbf{B}_s \times \mathbf{B}_t)$, which is formed by $\mathbf{F}_s$ and $q_t\mathbf{V}_t$ and $\mathbf{F}_t$ and $q_s\mathbf{V}_s$ (Chapter 9). After integrations over the field are carried out, one must examine the Poynting vector to isolate the term or terms that give the force experienced by $q_t\mathbf{V}_t$ and by $q_s\mathbf{V}_s$. Such terms will involve the velocities and charges carried by both particles. The sum of Eqs. (15.24), (15.25) defines the total energy density delivered to the field through the joint action of both particles. Both $q_s\mathbf{V}_s$ and $q_t\mathbf{V}_t$ will thereby experience a reactive force that must be identified. The negative of the reactive forces will then be regarded as the force on the particles. The force on $q_t\mathbf{V}_t$ depends on both $q_t\mathbf{V}_t$ and $q_s\mathbf{V}_s$. Likewise the force on $q_s\mathbf{V}_s$ also depends on $q_s\mathbf{V}_s$ and $q_t\mathbf{V}_t$. We will identify the reaction force on $q_t\mathbf{V}_t$ and then group it with the direct forces of $q_s\mathbf{V}_s$ on $q_t\mathbf{V}_t$ to obtain the net effective force on $q_t\mathbf{V}_t$. Likewise for the net effective force on $q_s\mathbf{V}_s$ as exerted by $q_t\mathbf{V}_t$.

$\mathbf{F}_s$ and $\mathbf{F}_t$ are defined by

$$\mathbf{F}_s = \mathbf{e}_0\mathbf{e}_1\frac{E_{sx}}{c} + \mathbf{e}_0\mathbf{e}_2\frac{E_{sy}}{c} + \mathbf{e}_0\mathbf{e}_3\frac{E_{sz}}{c} + B_{sz}\mathbf{e}_1\mathbf{e}_2 + B_{sy}\mathbf{e}_3\mathbf{e}_1 + B_{sx}\mathbf{e}_2\mathbf{e}_3 \tag{15.32}$$

$$\mathbf{F}_t = \mathbf{e}_0\mathbf{e}_1\frac{E_{tx}}{c} + \mathbf{e}_0\mathbf{e}_2\frac{E_{ty}}{c} + \mathbf{e}_0\mathbf{e}_3\frac{E_{tz}}{c} + B_{tz}\mathbf{e}_1\mathbf{e}_2 + B_{ty}\mathbf{e}_3\mathbf{e}_1 + B_{tx}\mathbf{e}_2\mathbf{e}_3 \tag{15.33}$$

To obtain the net force density $\mathbf{F}_T$ being delivered to the field from the above, form the sum

$$\mathbf{F}_T = \mathbf{F}_{ts}^{\mathrm{indir}} + \mathbf{F}_{st}^{\mathrm{indir}} \tag{15.34}$$

To obtain the equation for the vector force density in the field contributed jointly by $q_s\mathbf{V}_s$ and $q_t\mathbf{V}_t$, the following terms must be contracted. That contraction must be carried out is denoted by the symbol C.

$$\overset{\mathrm{Vec}}{\mathbf{F}_T} = \mu_0 k C \Bigg[\overset{\mathrm{Vec}}{(\mathbf{F}_s \bullet \mathbf{V}_t'' + \mathbf{F}_t \bullet \mathbf{V}_s'')} \overset{\mathrm{Scalar}}{(q_s q_t + q_t q_s)}/2 + \overset{\mathrm{Triv}}{\left(\mathbf{F}_s \wedge \mathbf{V}_t'' + \mathbf{F}_t \wedge \mathbf{V}_s''\right)} \overset{\mathrm{Biv}}{(q_s q_t - q_t q_s)}/2 \Bigg] \tag{15.35}$$

The trivector density in the field contributed jointly by $q_s\mathbf{V}_s$ and $q_t\mathbf{V}_t$, before the terms are contracted, is given by the bracketed terms before contraction.

$$\overset{\text{Triv}}{\mathbf{F}_T} = \frac{\mu_0 k}{2} C \left[\overset{\text{Triv}}{(\mathbf{F}_s \wedge \mathbf{V}_t'' + \mathbf{F}_t \wedge \mathbf{V}_s'')} \overset{\text{Scalar}}{(q_s q_t + q_t q_s)} + \overset{\text{Vec}}{(\mathbf{F}_s \bullet \mathbf{V}_t'' + \mathbf{F}_t \bullet \mathbf{V}_s'')} \overset{\text{Biv}}{(q_s q_t - q_t q_s)} \right] \tag{15.36}$$

After the terms in Eqs. (15.35) and (15.36) are contracted, they give the net vector force of $q_s\mathbf{V}_s$ and $q_t\mathbf{V}_t$ on the field. We denote the sums in Eqs. (15.35, 15.36) with the subscript T. They have a formal structure like the direct force of q_s on q_t, as in which case, instead of $\mathbf{F}_s$ above we had $\mathbf{B}_s = \mu_0 (\mathbf{V}_s \times \mathbf{r}_{st})$.

Note that the contraction operation converts the (Triv)(Biv) term in Eq. (15.35) into a vector as it was designed to do. The same operation applied to the (Vec)(Biv) term in Eq. (15.36) converts it into a trivector as it must. We do not evaluate the actual magnitude of the force since we have no scale to ab initio relate it to. Its magnitude can only be inferred from the fact that its existence can eliminate the need for dark matter. For conventional Newtonian gravity the gravitational constant G establishes a scale.

15.5 Summary

To obtain the complete force of $q_s\mathbf{V}_s$ and $q_t\mathbf{V}_t$ on the field before contraction we must evaluate

$$\mathbf{F}_T = \mathbf{F}_s^{\text{indir}}(\text{com}) + \mathbf{F}_s^{\text{indir}}(\text{non-com}) + \mathbf{F}_t^{\text{indir}}(\text{com}) + \mathbf{F}_t^{\text{indir}}(\text{non-com})$$

The symbol (com) denotes the group $(q_t q_s + q_s q_t)/2$. (non-com) denotes $(q_t q_s - q_s q_t)/2$.

$$\mathbf{F}_s^{\text{indir}}(\text{com}) = (\mathbf{F}_t \bullet \mathbf{V}_s)(q_t q_s + q_s q_t)/2 \tag{15.37}$$

$$\mathbf{F}_s^{\text{indir}}(\text{non-com}) = (\mathbf{F}_t \wedge \mathbf{V}_s)(q_t q_s - q_s q_t)/2 \tag{15.38}$$

$$\mathbf{F}_t^{\text{indir}}(\text{com}) = (\mathbf{F}_t \bullet \mathbf{V}_t)(q_s q_t + q_t q_s)/2 \tag{15.39}$$

$$\mathbf{F}_t^{\text{indir}}(\text{non-com}) = (\mathbf{F}_t \wedge \mathbf{V}_t)(q_s q_t - q_t q_s)/2 \tag{15.40}$$

The expression to be used for $\mathbf{V}_s$ when calculating the above indirect forces is

$$\mathbf{V}_s = \left(\mathbf{e}_0 \text{div} \frac{\mathbf{E}_s}{c} - \frac{1}{c^2} \frac{\partial \mathbf{E}_s}{\partial t} + \text{curl} \mathbf{B}_s \right) \tag{15.41}$$

$$\left(-\mathbf{e}_0 \text{div} \mathbf{B}_s + \frac{\partial \mathbf{B}_s}{\partial t} + \text{curl} \mathbf{E}_s \right) = 0 \tag{15.42}$$

Likewise

$$\mathbf{V}_t = \left(\mathbf{e}_0 \mathrm{div}\frac{\mathbf{E}_t}{c} - \frac{1}{c^2}\frac{\partial \mathbf{E}_t}{\partial t} + \mathrm{curl}\mathbf{B}_t\right) \quad (15.43)$$

$$\left(-\mathbf{e}_0 \mathrm{div}\mathbf{B}_t + \frac{\partial \mathbf{B}_t}{\partial t} + \mathrm{curl}\mathbf{E}_t\right) = 0 \quad (15.44)$$

$q_s\mathbf{V}_s$ and $q_t\mathbf{V}_t$ jointly deliver force into the field which will result in a reactive force on $q_t\mathbf{V}_t$ and on $q_s\mathbf{V}_s$. We regard the negative of the reaction forces as effectively the force that the currents exert on each other. Therefore after completion of the integration we must identify the terms that correspond to the reactive force on $q_s\mathbf{V}_s$ and on $q_t\mathbf{V}_t$, a bivector, but is also, without the unit vectors, the electromagnetic field tensor in standard E and M.

Also when evaluating the indirect force, $\mathbf{B}_s$ has been replaced by $\mathbf{F}_s$ where

$$\begin{aligned}\mathbf{F}_s &= \mathbf{e}_0\mathbf{e}_1\frac{E_{sx}}{c} + \mathbf{e}_0\mathbf{e}_2\frac{E_{sy}}{c} + \mathbf{e}_0\mathbf{e}_3\frac{E_{sz}}{c} + \mathbf{e}_1\mathbf{e}_2 B_{sz} + \mathbf{e}_3\mathbf{e}_1 B_{sy} + \mathbf{e}_2\mathbf{e}_3 B_{sx} \\ &= \mathbf{e}_0\mathbf{e}_1\frac{E_{sx}}{c} + \mathbf{e}_0\mathbf{e}_2\frac{E_{y}}{c} + \mathbf{e}_0\mathbf{e}_3\frac{E_{sz}}{c} + \mathbf{e}_5\left(\mathbf{e}_0\mathbf{e}_1 B_{sx} + \mathbf{e}_0\mathbf{e}_2 B_{sy} + \mathbf{e}_0\mathbf{e}_3 B_{sz}\right) \\ &= \frac{\mathbf{E}_s}{c} + \mathbf{e}_5\mathbf{B}_s \qquad \mathbf{e}_5 = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \end{aligned} \quad (15.45)$$

$\mathbf{e}_5 = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ anticommutes with a vector and commutes with a bivector. $\mathbf{e}_5\mathbf{e}_5 = -1$.

A similar expression applies to $\mathbf{F}_t$ by changing the subscript s to t. Thus the total indirect force $\mathbf{F}^{\mathrm{indir},T}$ of $q_s\mathbf{V}_s$ and $q_t\mathbf{V}_t$ on the field is, where

$$\mathbf{F}^{\mathrm{indir},T} = \mathbf{F}^{\mathrm{indir},T}_{ts} + \mathbf{F}^{\mathrm{indir},T}_{st},$$

$$\overset{\text{vec}}{\mathbf{F}^{\mathrm{indir},\,T}_{ts}} = \mathbf{F}_s \bullet \overset{\text{vec}}{(\mathbf{V}_t)}\overset{\text{scalar}}{(q_s q_t + q_t q_s)}/2 + \mathbf{F}_s \wedge \overset{\text{triv}}{(\mathbf{V}_t)}\overset{\text{biv}}{(q_s q_t - q_t q_s)}/2 \quad (15.46)$$

$$\overset{\text{vec}}{\mathbf{F}^{\mathrm{indir},T}_{st}} = \mathbf{F}_t \bullet \overset{\text{vec}}{(\mathbf{V}_s)}\overset{\text{scalar}}{(q_t q_s + q_s q_t)}/2 + \mathbf{F}_t \wedge \overset{\text{triv}}{(\mathbf{V}_s)}\overset{\text{biv}}{(q_t q_s - q_s q_t)}/2 \quad (15.47)$$

The forces appearing in Eqs. (15.46), (15.47) are force densities. As previously stated, the sum contains the time derivative of the Poynting linear momentum vector. To obtain the final result for the total force on the field will require an integration of all terms over all space followed by differentiation with respect to time. One must then isolate the terms that may be identified as forces acting on q_t and q_s. When $q_s\mathbf{V}_s$ and $q_t\mathbf{V}_t$ exert a force in the field they both experience a reactive force. We identify the negative of the reactive force on q_t as the indirect force of q_s on q_t. Likewise the negative of the reactive force on $q_s\mathbf{V}_s$ is called the indirect force of $q_t\mathbf{V}_t$ on $q_s\mathbf{V}_s$, since there would be no force on $q_s\mathbf{V}_s$ from these terms if $q_t\mathbf{V}_t$ were zero.

$q_t\mathbf{v}_t$ and $q_s\mathbf{v}_s$ will also contribute angular momentum density to the field. Forces for this will be described later and must be included to obtain the net effective force of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$ and vice versa.

Note that the direct forces involve the quantity $\mathbf{B}_s$, which, in Eq. (15.19) specifies the field specifically at a particular position of $q_t\mathbf{V}_t$. $\mathbf{B}_s$ replaced by $\mathbf{F}_s$ describes the fields throughout space. Thus when we want the force exerted directly by $q_s\mathbf{V}_s$ on a particular current element $q_t\mathbf{V}_t$ we employ $\mathbf{B}_s = \mu_0 q_s (\mathbf{V}_s \wedge \mathbf{r}_{st})/r^3$ where $\mathbf{r}_{st}$ is the distance from $q_s\mathbf{V}_s$ to $q_t\mathbf{V}_t$. On the other hand $\mu_0 q_s (\mathbf{V}_s \wedge \mathbf{r}_{st})/r^3$ is also the source for the electromagnetic field throughout space. In that case, we employ Eqs. (15.41) and (15.42) for velocities ("field velocities") and designate the field components by the coefficients in the bivector $\mathbf{F}_s$, Eq. (15.32), which replaces $\mathbf{B}_s$, Eq. (15.19), (15.20). In conventional language $\mathbf{F}_s$ is the anti-symmetric electromagnetic field tensor. We will use it to obtain differential equations for the field components. We can also use it to deduce the way the fields are propagated when generated by time varying sources. The way in which the fields interact with matter is specified by boundary conditions, rather than through the point to point interaction described by Eq. (15.19).

We now display in the same equation, terms that contain the source of the total vector force on $q_t\mathbf{V}_t$ both direct and indirect and which involve the scalar sum $(q_s q_t + q_t q_s)/2$.

$$\mathbf{F}_{ts}(\text{vec, com}) = \left[\mu_0 k (\mathbf{V}_s \wedge \mathbf{r}_{st}) \underset{\text{Vec}}{\overset{\text{dir}}{\bullet}} \mathbf{V}_t + \mathbf{F}_s \underset{\text{Vec}}{\overset{\text{indir}}{\bullet}} \mathbf{V}_t\right] \underset{\text{Scalar}}{(q_s q_t + q_t q_s)/2} \tag{15.48}$$

Similarly the single equation involving $(q_t q_s + q_s q_t)/2$ that contains all terms responsible for the force on $q_s\mathbf{V}_s$.

$$\mathbf{F}_{st}(\text{vec, com}) = \left[\mu_0 k (\mathbf{V}_t \wedge \mathbf{r}_{ts}) \underset{\text{Vec}}{\overset{\text{dir}}{\bullet}} \mathbf{V}_s + \mathbf{F}_t \underset{\text{Vec}}{\overset{\text{indir}}{\bullet}} \mathbf{V}_s\right] \underset{\text{Scalar}}{(q_t q_s + q_s q_t)/2} \tag{15.49}$$

Eqs. (15.48), (15.49) do not require contraction since $(q_t q_s + q_s q_t)/2$ is always a scalar. The first term in Eq. (15.48) is the direct force of $q_s\mathbf{V}_s$ on $q_t\mathbf{V}_t$ and is complete as it stands. The second two terms are the force density delivered to the field by $\mathbf{F}_s \bullet \mathbf{V}_t$. To obtain forces, they must be integrated over all space and differentiated with respect to time. When this is done we will identify terms that turn out to be responsible for the force experienced by $q_t\mathbf{V}_t$.

One must add an additional direct and indirect vector force when the "charges" do not commute. To denote this force, we use the symbol $\mathbf{F}_{ts}^{\text{non-com}}$. Non-com stands for non-commuting.

$$\begin{aligned}&\mathbf{F}_{ts}\left(\text{vec, non-com}\right)\\ =\ &\left[\mu_0 k(\mathbf{V}_s \wedge \mathbf{r}_{st}) \overset{\text{dir}}{\underset{\text{Triv}}{\wedge}} \mathbf{V}_t + \mathbf{F}_s \overset{\text{ind}}{\underset{\text{Triv}}{\wedge}} \mathbf{V}_t\right] \left(q_s q_t \underset{\text{Biv}}{-} q_t q_s\right)/2 \qquad (15.50)\\ &\mathbf{F}_{st}\left(\text{vec, non-com}\right)\\ =\ &\left[\mu_0 k(\mathbf{V}_t \wedge \mathbf{r}_{ts}) \overset{\text{dir}}{\underset{\text{Triv}}{\wedge}} \mathbf{V}_s + \mathbf{F}_t \overset{\text{ind}}{\underset{\text{Triv}}{\wedge}} \mathbf{V}_s +\right] \left(q_t q_s \underset{\text{Biv}}{-} q_s q_t\right)/2 \qquad (15.51)\end{aligned}$$

Equations Eqs. (15.50), (15.51) require contraction.

The total vector force, $\mathbf{F}_{ts}^{T}\left(\text{vec}\right)$, delivered to the field is the sum of equations Eqs. (15.48), (15.49)

$$\begin{aligned}&\mathbf{F}_{ts}^{T}\left(\text{vec}\right) + \mathbf{F}_{st}\left(\text{vec}\right) = \qquad (15.52)\\ &\mathbf{F}_{ts}^{em}\left(\text{vec, com}\right) + \mathbf{F}_{ts}^{N}\left(\text{vec, non-com}\right)\\ &+\mathbf{F}_{st}^{em}\left(\text{vec, com}\right) + \mathbf{F}_{st}^{N}\left(\text{vec, non-com}\right)\end{aligned}$$

Note again that the combination $\mathbf{F}_s \wedge (\mathbf{V}_t + \mathbf{e}_5 \mathbf{U}_t)$ in Eq. (15.52) is a trivector as are the second and third terms in Eq. (15.50). In Eq. (15.50) $\left(q_s q_t - q_t q_s\right)/2$ is a bivector. After contraction Eqs. (15.46, 15.50) yield a vector. Contraction has been defined such that this is so.

In addition to the above vector forces, there is a trivector effect of $q_s \mathbf{V}_s$ on $q_t \mathbf{V}_t$. The direct trivector is given by

$$\mathbf{F}_{ts}^{\text{triv dir}}\left(\text{com}\right) = \mu_0 k(\mathbf{B}_s \underset{\text{triv}}{\wedge} \mathbf{V}_t)(q_s q_t \underset{\text{scalar}}{+} q_t q_s)/2 \qquad (15.53)$$

where, as before $\mathbf{B}_s = \mathbf{V}_s \wedge \mathbf{r}_{st}$

In Eq. (15.53), $k = 1/4\pi r^2$ is the spatial spreading factor for a massless field. For a massive field $1/4\pi r^2$ is replaced by

$$k' = \frac{1}{4\pi} \frac{d}{dr} \left(\frac{e^{-\mu r}}{r}\right)$$

This replacement may be made in principle in all force laws discussed herein. Note that the same combination $\mathbf{F}_s \wedge \mathbf{V}_t + \mathbf{e}_5 \mathbf{F}_s \wedge \mathbf{U}_t$ occurs in Eq. (15.50) multiplied by $\left(q_s q_t - q_t q_s\right)/2$.

If an indirect trivector exists, its structure would be given by

$$\mathbf{F}_{ts}^{\text{indir}}\,(\text{triv, com}) = \mathbf{F}_s \wedge (\mathbf{V}_t + \mathbf{e}_5\mathbf{U}_t)\,(q_s q_t + q_t q_s)\,/2 \tag{15.54}$$

Eq. (15.54) would give the force imparted to the field by such a force, where again $\mathbf{F}_s$ is given by

$$\mathbf{F}_s = \frac{E_{sx}}{c}\mathbf{e}_0\mathbf{e}_1 + \frac{E_{sy}}{c}\mathbf{e}_0\mathbf{e}_2 + \frac{E_{sz}}{c}\mathbf{e}_0\mathbf{e}_3 + B_{sz}\mathbf{e}_1\mathbf{e}_2 + B_{sy}\mathbf{e}_3\mathbf{e}_1 + B_{sx}\mathbf{e}_2\mathbf{e}_3 \tag{15.55}$$

with $\mathbf{V}_t$ and $\mathbf{U}_t$ given by Eqs. (15.46) and (15.47). As we will see later, an indirect trivector interaction, Eq. (15.54), does not exist. Since we do not know this at this stage, we carry along the structure it would have and later verify that its value is zero.

Adding Eqs. (15.53) and (15.54), the total trivector interaction of $q_s\mathbf{V}_s$ on $q_t\mathbf{V}_t$ is given by Eq. (15.56) (no contraction necessary). The notation $\mathbf{F}_{ts}\,(\text{triv, com})$ means that q_s, q_t that describe the property, commute. This is clear, of course, from the multiplier $(q_s q_t + q_t q_s)\,/2$.

$$\begin{aligned}&\mathbf{F}_{ts}\,(\text{triv, com})\\ =\;& \mu_0 k \left[\underset{\text{Triv}}{\overset{\text{dir}}{(\mathbf{V}_s \wedge \mathbf{r}_{st}) \wedge \mathbf{V}_t}} + \overset{\text{ind}}{\mathbf{F}_s \wedge \mathbf{V}_t}\right] \underset{\text{Scalar}}{(q_s q_t + q_t q_s)/2}\end{aligned} \tag{15.56}$$

$$(q_s q_t + q_t q_s)\,/2 = \left(-c^2 + v_s' v_t'\right) \quad \text{when} \quad q_s = (e_0 c + v_s')\,, \qquad q_t = (e_0 c + v_t')$$

To which we must add, when the "charges" do not commute (contraction necessary)

$$\begin{aligned}&\mathbf{F}_{ts}\,(\text{triv, non-com})\\ =\;& \mu_0 k \left[\overset{\text{dir}}{(\mathbf{V}_s \wedge \mathbf{r}_{st}) \bullet \mathbf{V}_t} + \overset{\text{ind}}{\mathbf{F}_s \bullet \mathbf{V}_t}\right] \underset{\text{Biv}}{(q_s q_t - q_t q_s)/2}\end{aligned} \tag{15.57}$$

The total trivector interaction between $q_s\mathbf{V}_s$ and $q_t\mathbf{V}_t$ is given by the sum of equations Eqs. (15.56) and (15.57).

$$\mathbf{F}_{ts}^{T}\,(\text{triv}) = \mathbf{F}_{ts}\,(\text{triv}, \text{com}) + \mathbf{F}_{ts}\,(\text{triv}, \text{non-com}) \tag{15.58}$$

In each case $q_s\mathbf{V}_s$ describes the motion of a space-time multivector q_s and $q_t\mathbf{V}_t$ represents the same for the space-time multivector q_t. Equation (15.50) gives the vector force that q_s exerts on q_t. Equation (15.58) gives the trivector interaction of q_s on q_t.

These results indicate that every vector force in nature may have a trivector companion. The electromagnetic electric force is a special case: the interacting electric charges q_s and q_t are scalars.

If the interaction source is a vector but with a magnitude and range governed by a Yukawa factor involving smaller mass(es), the resultant longer range force is the strong force.

The above formalism also applies to anti-matter.

As seen, both the vector force and the trivector have an additional contribution when the multivectors q_s and q_t do not commute. When the vector force is mediated by very heavy particles, the direct force is easily modified by simply replacing $1/4\pi r^2$ by $\frac{d}{dr}\left(\frac{e^{-\mu r}}{r}\right)$. However the field tensor, Eq. (15.55) must also be modified to conform to the Proca field which describes massive particle exchange. We omit a discussion of those changes here.

15.6 Mixed Multivector Interactions

For the interaction of multivectors of different kinds, label one multivector with a prime, for example q_t'. The source multivector q_s is unprimed; once again, roles are interchangeable. The combination $\left(q_t' q_s + q_s q_t'\right)/2$ is no longer a scalar. The structure depends on the particular multivectors q_t' and q_s. The multivectors resulting from various q_t' and q_s are tabulated in Table A. For example, if q_t' is electric charge (electromagnetism) and q_s is a 4-momentum vector P_s (gravity), then the effective charge is $\left(q_t' P_s + P_s q_t'\right)/2$, a vector quantity.

We will not pursue the details of different multivector interactions here but the available input for doing so is contained in the Tables. In connection with negative gravity, we add the following obvious statement. A constant force f, acting on a mass m, produces a constant acceleration given by $f = -ma$. The velocity, after time t, is

$$v = -mat$$

Thus, v changes linearly with time. In general, an inverse r^2 factor must be included. f is produced by another mass M so the force is between two masses.