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ALTERNATE DERIVATION OF THE ANGULAR MOMENTUM EQUATION FOR INTERACTING PARTICLES

13.1 Alternate Derivation of the Angular Momentum Equation for Interacting Particles

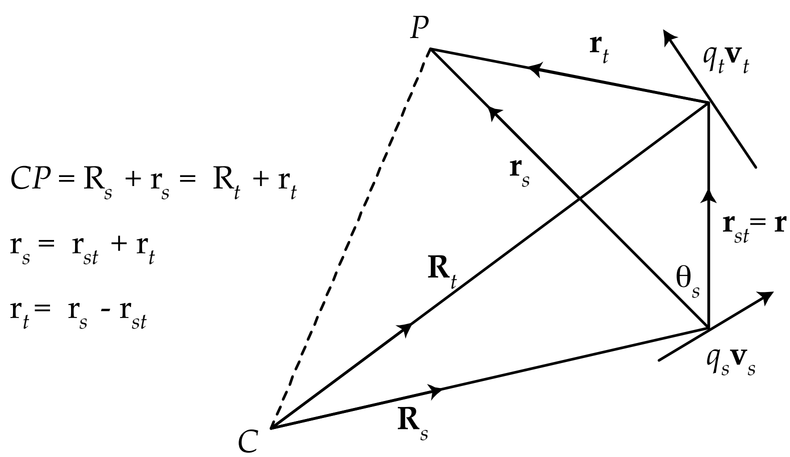

Fig. 13.

The electromagnetic linear momentum density, $\mathbf{g}_{T\ell}$, delivered to the field at point P by $q_s\mathbf{v}_s$ and $q_t\mathbf{v}_t$ is

$$\begin{aligned}\mathbf{g}_{T\ell} &= \varepsilon_0 \left(\mathbf{E}_s + \mathbf{E}_t\right) \times \left(\mathbf{B}_s + \mathbf{B}_t\right) \\ &= \varepsilon_0 \left(\mathbf{E}_s \times \mathbf{B}_s + \mathbf{E}_s \times \mathbf{B}_t + \mathbf{E}_t \times \mathbf{B}_s + \mathbf{E}_t \times \mathbf{B}_t\right) \qquad (13.1)\end{aligned}$$

The mutual linear electromagnetic momentum density, $\mathbf{g}_\ell$, at a field point P is

$$\mathbf{g}_\ell = \varepsilon_0 \left(\mathbf{E}_s \times \mathbf{B}_t + \mathbf{E}_t \times \mathbf{B}_s\right) = \mathbf{g}_{\ell 1} + \mathbf{g}_{\ell 2} \tag{13.2}$$

$$\mathbf{B}_s = \frac{\mu_0 q_s}{4\pi}\frac{\mathbf{v}_s \times \mathbf{r}_s}{r_s^3} = \frac{q_s}{4\pi\varepsilon_0 c^2}\frac{\mathbf{v}_s \times \mathbf{r}_s}{r_s^3} \qquad \mathbf{E}_s = \frac{q_s}{4\pi\varepsilon_0}\frac{\mathbf{r}_s}{r_s^3} \tag{13.3}$$

$$\mathbf{B}_t = \frac{q_t}{4\pi\varepsilon_0 c^2}\frac{\mathbf{v}_t \times \mathbf{r}_t}{r_t^3} \qquad \mathbf{E}_t = \frac{q_t}{4\pi\varepsilon_0}\frac{\mathbf{r}_t}{r_t^3} \tag{13.4}$$

$\mathbf{r}_s$ and $\mathbf{r}_t$ are distances from q_s and q_t, respectively, to the field point P, Fig. 13.1. Subscripts $\ell 1$ and $\ell 2$ refer to the linear momenta of the first and second terms respectively in Eq. (13.2).

The strength of $\mathbf{B}_s$ and $\mathbf{B}_t$ goes as $1/c^2$. Velocity correction to $\mathbf{E}_s$ and $\mathbf{E}_t$ will be required to include terms in $1/c^2$.

In general, for arbitrary multivectors q_s and q_t, the linear momentum density delivered to the field is:

$$\mathbf{g}_\ell = \frac{\varepsilon_0 \left[\mathbf{r}_s \times (\mathbf{v}_t \times \mathbf{r}_t) + \mathbf{r}_t \times (\mathbf{v}_s \times \mathbf{r}_s)\right]}{\left(4\pi\varepsilon_0\right)^2 c^2}\frac{(q_s q_t + q_t q_s)}{2} = \mathbf{g}_{\ell 1} + \mathbf{g}_{\ell 2}$$

let

$$q_t q_s \varepsilon_0 / \left(4\pi\varepsilon_0\right)^2 c^2 = k$$

$$\underset{\mathbf{E}_s}{\mathbf{r}_s} \times (\underset{\mathbf{B}_t}{\mathbf{v}_t \times \mathbf{r}_t}) + \underset{\mathbf{E}_t}{\mathbf{r}_t} \times (\underset{\mathbf{B}_s}{\mathbf{v}_s \times \mathbf{r}_s}) = \underbrace{\mathbf{v}_t \left(\mathbf{r}_s \cdot \mathbf{r}_t\right) - \mathbf{r}_t \left(\mathbf{r}_s \cdot \mathbf{v}_t\right)}_{\mathbf{E}_s \times \mathbf{B}_t} + \underbrace{\mathbf{v}_s \left(\mathbf{r}_t \cdot \mathbf{r}_s\right) - \mathbf{r}_s \left(\mathbf{r}_t \cdot \mathbf{v}_s\right)}_{\mathbf{E}_t \times \mathbf{B}_s}$$

Relative to the point C, the position vector of the field point P is

$$CP = \mathbf{R}_s + \mathbf{r}_s = \mathbf{R}_t + \mathbf{r}_t \tag{13.5}$$

The total angular momentum is

$$\mathbf{G} = \int \mathbf{g}_a d\tau = \int \left(\mathbf{R}_s + \mathbf{r}_s\right) \mathbf{g}_{\ell 1} d\tau + \int \left(\mathbf{R}_t + \mathbf{r}_t\right) \mathbf{g}_{\ell 2} d\tau \tag{13.6}$$

using $\mathbf{R}_s + \mathbf{r}_s = \mathbf{R}_t + \mathbf{r}_t$.

The integration must be over all space. We have already integrated the linear momentum density $\mathbf{g}_\ell$ over all space to obtain the total linear momentum, $\mathbf{G}_\ell$. It is

$$\mathbf{G}_\ell = \int \mathbf{g}_\ell d\tau = \frac{1}{4\pi\varepsilon_0 c^2}\frac{1}{2}\left[\overbrace{\underbrace{\frac{\mathbf{v}_t}{r} + \frac{\left(\mathbf{v}_t \cdot \mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}}_{\ell 1}}^{\varepsilon_0 \int \mathbf{E}_s \times \mathbf{B}_t d\tau} + \overbrace{\underbrace{\frac{\mathbf{v}_s}{r} + \frac{\left(\mathbf{v}_s \cdot \mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}}_{\ell 2}}^{\varepsilon_0 \int \mathbf{E}_t \times \mathbf{B}_s d\tau}\right](q_s q_t + q_t q_s)/2 \tag{13.7}$$

We now evaluate the second term in Eq. (13.6) by integrating it over all space. That is, we evaluate

$$\int (\mathbf{R}_s + \mathbf{r}_s) \times \mathbf{g}_{ts\ell}^{\varepsilon_0(\mathbf{E}_t \times \mathbf{B}_s)} d\tau =$$

$$\frac{q_t q_s \varepsilon_0}{(4\pi\varepsilon_0)^2 c^2} \int (\mathbf{R}_s + \mathbf{r}_s) \times \frac{\overbrace{[\mathbf{v}_s (\mathbf{r}_s \cdot \mathbf{r}_t) - \mathbf{r}_s (\mathbf{r}_t \cdot \mathbf{v}_s)]}^{g_{\ell 2} = \varepsilon_0 \int \mathbf{E}_t \times \mathbf{B}_s d\tau}}{r_s^3 r_t^3} d\tau \quad (q_s q_t + q_t q_s)/2 \tag{13.8}$$

For gravity, multiply Eqs. (13.7) and (13.8) by $(q_s q_t + q_t q_s)/2 = (P_s P_t + P_t P_s)/2$. We have already evaluated the two integrals

$$\int \mathbf{g}_{ts\ell}^{\varepsilon_0(\mathbf{E}_t \times \mathbf{B}_s)} d\tau = \frac{q_s q_t}{4\pi\varepsilon_0} \frac{1}{2c^2} \left(\frac{\mathbf{v}_s}{r} + \frac{\mathbf{v}_s \cdot \mathbf{r}_{st}}{r^3} \right) \tag{13.9}$$

and

$$\int \mathbf{g}_{st\ell}^{\varepsilon_0(\mathbf{E}_s \times \mathbf{B}_t)} d\tau = \frac{q_s q_t}{4\pi\varepsilon_0} \frac{1}{2c^2} \left(\frac{\mathbf{v}_t}{r} + \frac{\mathbf{v}_t \cdot \mathbf{r}_{st}}{r^3} \right) \tag{13.10}$$

Thus

$$\int (\mathbf{R}_s + \mathbf{r}_s) \times \mathbf{g}_{ts\ell}^{\varepsilon_0(\mathbf{E}_t \times \mathbf{B}_s)} d\tau =$$

$$\frac{q_t q_s}{4\pi\varepsilon_0} \mathbf{R}_s \times \frac{1}{2c^2} \left(\frac{\mathbf{v}_s}{r} + \frac{(\mathbf{v}_s \cdot \mathbf{r}_{st}) \mathbf{r}_{st}}{r^3} \right) + \frac{q_t q_s}{(4\pi\varepsilon_0)^2} \frac{\varepsilon_0}{c^2} \int \frac{(\mathbf{r}_s \times \mathbf{v}_s)(\mathbf{r}_s \cdot \mathbf{r}_t)}{r_s^3 r_t^3} d\tau \tag{13.11}$$

In Eq. (13.11), $\mathbf{r}_s \cdot \mathbf{r}_t = (r_s^2 + r_t^2 - r_{st}^2)/2$.

We now evaluate the second term in Eq. (13.11).

$$\frac{q_t q_s}{(4\pi\varepsilon_0)^2} \frac{\varepsilon_0}{c^2} \int \frac{(\mathbf{r}_s \times \mathbf{v}_s)(r_s^2 + r_t^2 - r_{st}^2)/2}{r_s^3 r_t^3} d\tau \tag{13.12}$$

In general, replace $q_s q_t$ by $(q_s q_t + q_t q_s)/2$.

For gravity, $(q_s q_t + q_t q_s)/2 = (P_s P_t + P_t P_s)/2$ where $P_s = c\mathbf{e}_0 + \mathbf{v}_s$ and $P_t = c\mathbf{e}_0 + \mathbf{v}_t$.

$$\mathbf{r}_s \times \mathbf{v}_s = r_s \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \mathbf{v}_{sx} & \mathbf{v}_{sy} & \mathbf{v}_{sz} \end{vmatrix}$$

$$\begin{aligned} \mathbf{r}_s \times \mathbf{v}_s &= r_s [\mathbf{e}_1 (v_{sz} \sin\theta\cos\varphi - v_{sy}\cos\theta) + \mathbf{e}_2 (v_{sx}\cos\theta - v_{sz}\sin\theta\cos\varphi) \\ &\quad + \mathbf{e}_3 (v_{sy}\sin\theta\cos\varphi - v_{sx}\sin\theta\sin\varphi) \end{aligned}$$

$$\begin{aligned}(\mathbf{r}_s \times \mathbf{v}_s)(\mathbf{r}_t \cdot \mathbf{r}_s) &= r_s \left[\mathbf{e}_1 \left(v_{sz} \sin\theta \cos\varphi - \underline{v_{sy} \cos\theta}\right)\right] \\ &\quad + \mathbf{e}_2 \left[\underline{v_{sx} \cos\theta} - v_{sz} \sin\theta \cos\varphi\right] \\ &\quad + \mathbf{e}_3 \left[v_{sy} \sin\theta \cos\varphi - v_{sx} \sin\theta \sin\varphi\right] \frac{(r_s^2 + r_t^2 - r^2)}{2} \end{aligned} \quad (13.13)$$

$d\tau = (r_s r_t / r_{st})\, dr_t dr_s d\varphi = r_{st}^3 \rho_s \rho_t d\rho_s d\rho_t d\varphi.$

When Eq. (13.13) is integrated over φ, only the underlined terms are non-zero. The result is

$$(\mathbf{r}_s \times \mathbf{v}_s)(\mathbf{r}_t \cdot \mathbf{r}_s) = 2\pi r_s \left[-\mathbf{e}_1 v_{sy} \cos\theta + \mathbf{e}_2 v_{sx} \cos\theta\right] \left(r_s^2 + r_t^2 - r_{st}^2\right)/2$$

In dimensionless variables, $r_s = r_{st}\rho_s, \qquad r_t = r_{st}\rho_t$

$$\begin{aligned}(\mathbf{r}_s \times \mathbf{v}_s)(\mathbf{r}_t \cdot \mathbf{r}_s) &= -2\pi r_{st}^2 \left[\mathbf{e}_1 v_{sy} - \mathbf{e}_2 v_{sx}\right] r_s \cos\theta \left(\rho_s^2 + \rho_t^2 - 1\right)/2 \\ &= -2\pi r_{st}^3 \left(\mathbf{v}_s \times \mathbf{e}_3\right) \rho_s \cos\theta \left(\rho_s^2 + \rho_t^2 - 1\right)/2 \end{aligned} \quad (13.14)$$

Now evaluate

$\rho_s \cos\theta \left[(\rho_s^2 - 1) + \rho_t^2\right], \qquad \cos\theta = \left[(1 + \rho_s^2) - \rho_t^2\right]/2\rho_s$

First evaluate $\rho_s \cos\theta \,(\rho_s^2 - 1)$

$$\begin{aligned} &\int_0^1 \int_{1-\rho_s}^{1+\rho_s} \frac{\rho_s \cos\theta \,(\rho_s^2 - 1)}{\rho_s^2 \rho_t^2} d\rho_s d\rho_t = \int_0^1 \int_{1-\rho_s}^{1+\rho_s} \frac{(\rho_s^2 - 1)\left[(1 + \rho_s^2) - \rho_t^2\right]}{\rho_s^2 \rho_t^2 2\rho_s} d\rho_s d\rho_t \\ =& \int_0^1 \int_{1-\rho_s}^{1+\rho_s} \left[\frac{(\rho_s^2 - 1)(\rho_s^2 + 1)}{2\rho_s^2 \rho_t^2} - \frac{(\rho_s^2 - 1)\rho_t^2}{2\rho_s^2 \rho_t^2}\right] d\rho_s d\rho_t \\ =& \int_0^1 \left[\frac{(\rho_s^2 - 1)(\rho_s^2 + 1)\, 2\rho_s}{2\rho_s^2 (1 - \rho_s^2)} - \frac{(\rho_s^2 - 1)\, 2\rho_s}{2\rho_s^2}\right] d\rho_s \\ =& \int_0^1 \left[\frac{-(\rho_s^2 + 1)\, 2\rho_s - 2\rho_s^2 + 2\rho_s}{2\rho_s^2}\right] d\rho_s = \int_0^1 \left[\frac{-2\rho_s^3 - 2\rho_s - 2\rho_s^3 + 2\rho_s}{2\rho_s^2}\right] d\rho_s \\ =& \int_0^1 \frac{-4\rho_s^3}{2\rho_s^2} d\rho_s = \int_0^1 -2\rho_s d\rho_s = \left[-\rho_s^2\right]_0^1 = -1 \end{aligned} \quad (13.15)$$

Now evaluate

$$\int_0^1\int_{1-\rho_s}^{1+\rho_s}\frac{\rho_s\cos\theta\rho_t^2}{\rho_s^2\rho_t^2}d\rho_t d\rho_s = \int_0^1\int_{1-\rho_s}^{1+\rho_s}\frac{1}{\rho_s^2}\left[\frac{(1+\rho_s^2)-\rho_t^2}{2\rho_t^2}\right]d\rho_t d\rho_s$$

$$= \int_0^1\int_{1-\rho_s}^{1+\rho_s}\frac{1}{\rho_s^2}\frac{1}{2}\left[\frac{1+\rho_s^2}{\rho_t^2}-1\right]d\rho_t d\rho_s$$

$$= \int_0^1\frac{1}{\rho_s^2 2}\left[\frac{(1+\rho_s^2)}{1-\rho_s^2}-\frac{2\rho_s}{1-\rho_s^2}\right]d\rho_s = \int_0^1\frac{2}{\rho_s^2 2}\left[\frac{\rho_s+\rho_s^3-\rho_s+\rho_s^3}{1-\rho_s^2}\right]d\rho_s$$

$$= \int_0^1\frac{1}{\rho_s^2}\frac{2\rho_s^3 d\rho_s}{1-\rho_s^2} = \int_0^1\frac{2\rho_s d\rho_s}{1-\rho_s^2} \tag{13.16}$$

$$\therefore \quad \int\limits_0^1\int\limits_{1-\rho_s}^{1+\rho_s}\cos\theta\left[\left(\rho_s^2-1\right)+\rho_t^2\right]\frac{d\rho_s d\rho_t}{\rho_s^2\rho_t^2} = -1+1=0 \tag{13.17}$$

Now evaluate

$$\int_1^\infty\int_{\rho_s-1}^{\rho_s+1}\frac{\rho_s\cos\theta\left(\rho_s^2-1\right)}{\rho_s^2\rho_t^2}d\rho_s d\rho_t$$

$$= \int_1^\infty\int_{\rho_s-1}^{\rho_s+1}\left[\frac{\left(\rho_s^2-1\right)\left(\rho_s^2+1\right)}{2\rho_s^2\rho_t^2}-\frac{\left(\rho_s^2-1\right)\rho_t^2}{2\rho_s^2\rho_t^2}\right]d\rho_s d\rho_t$$

$$= \int_1^\infty\left[\frac{\left(\rho_s^2-1\right)\left(\rho_s^2+1\right)2}{2\rho_s^2\left(\rho_s^2-1\right)}-\frac{\left(\rho_s^2-1\right)}{2\rho_s^2}\right]d\rho_s$$

$$= \int_1^\infty\left[\frac{\left(\rho_s^2+1\right)}{\rho_s^2}-\frac{\left(\rho_s^2-1\right)}{\rho_s^2}\right]d\rho_s = \int_1^\infty\frac{2}{\rho_s^2}d\rho_s = -2\int_1^\infty d\left(\frac{1}{\rho_s}\right)$$

$$= \left[-2\frac{1}{\rho_s}\right]_1^\infty = 2$$

$$\int_1^\infty\int_{\rho_s-1}^{\rho_s+1}\frac{\rho_s\cos\theta\rho_t^2}{\rho_s^2\rho_t^2}d\rho_s d\rho_t = \iint\frac{1}{\rho_s^2}\frac{\left(1+\rho_s^2-\rho_t^2\right)}{2}d\rho_t d\rho_s$$

$$= \int_1^\infty\int_{\rho_{s-1}}^{\rho_{s+1}}\frac{1}{\rho_s^2}\frac{1}{2}\left[\frac{(1+\rho_s^2)}{\rho_t^2}-1\right]d\rho_t d\rho_s = \int_1^\infty\frac{1}{\rho_s^2 2}\left[\frac{(1+\rho_s^2)\,2}{\rho_s^2-1}-2\right]d\rho_s$$

$$= \int_1^\infty\frac{2}{\rho_s^2 2}\left[1+\rho_s^2-\rho_s^2+1\right]d\rho_s$$

$$= \int_1^\infty\frac{2}{\rho_s^2}d\rho_s = 2\left[-\frac{1}{\rho_s}\right]_1^\infty = -2$$

$$\therefore \int_1^\infty \int_{\rho_s-1}^{\rho_s+1} \cos\theta \left[\left(\rho_s^2-1\right)+\rho_t^2\right] \frac{d\rho_s d\rho_t}{\rho_s^2\rho_t^2} = 2-2=0$$

Then, finally Eq. (13.16) for the total angular momentum about C is

$$\mathbf{G}_a = \frac{q_t q_s}{4\pi\varepsilon_0 c^2}\frac{1}{2}\left\{\mathbf{R}_t \times \left[\frac{\mathbf{v}_s}{r} + \frac{(\mathbf{v}_s\cdot\mathbf{r}_{st})\,\mathbf{r}_{st}}{r^3}\right] + \mathbf{R}_s \times \left[\frac{\mathbf{v}_t}{r} + \frac{(\mathbf{v}_t\cdot\mathbf{r}_{st})\,\mathbf{r}_{st}}{r^3}\right]\right\} \tag{13.18}$$

To quote Page and Adams (1945, 144), "we see that the portion of the linear momentum involving the velocity $\mathbf{v}_s$ of particle q_s is to be considered as located at the position of particle q_t which has velocity $\mathbf{v}_t$ and vice versa."

13.2 Proca Equations as Modified Maxwell Equations

For a massive field we apply

$$(\Box - M) \quad \text{to} \quad (\mathbf{A} + \Box\theta) \tag{13.19}$$

$$\begin{aligned}
\Box - M(\mathbf{A}+\Box\theta) &= \Box\mathbf{A}+\Box(\Box\theta) - M\mathbf{A} - M\Box\theta \\
&= \Box\wedge\mathbf{A}+\Box\bullet\mathbf{A}+\Box^2\theta - M\mathbf{A} - M\Box\theta \\
&= \mathbf{F}+\Box\bullet\mathbf{A}+\Box^2\theta - M(\mathbf{A}+\Box\theta)
\end{aligned} \tag{13.20}$$

Where, as for E and M

$$\begin{aligned}
\mathbf{F} &= \Box\mathbf{A} = \mathbf{e}_0\mathbf{e}_1\left(-\frac{\partial A_1}{c\partial t} - \frac{\partial A_0}{\partial x}\right) + \mathbf{e}_0\mathbf{e}_2\left(-\frac{\partial A_2}{c\partial t} - \frac{\partial A_0}{\partial y}\right) + \mathbf{e}_0\mathbf{e}_3\left(-\frac{\partial A_3}{c\partial t} - \frac{\partial A_0}{\partial z}\right) \\
&\quad +\mathbf{e}_1\mathbf{e}_2\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) + \mathbf{e}_3\mathbf{e}_1\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \mathbf{e}_2\mathbf{e}_3\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) \\
&= (\mathbf{e}_0\mathbf{e}_1 E_x + \mathbf{e}_0\mathbf{e}_2 E_y + \mathbf{e}_0\mathbf{e}_3 E_z)/c + \mathbf{e}_1\mathbf{e}_2 B_x + \mathbf{e}_3\mathbf{e}_1 B_y + \mathbf{e}_2\mathbf{e}_3 B_z
\end{aligned}$$

As in electromagnetism, we have put

$$\Box\bullet\mathbf{A}+\Box^2\theta = 0 \quad \text{or} \quad \Box\bullet\mathbf{A} = 0 \quad \text{and} \quad \Box^2\theta = 0$$

Then

$$(\Box - M)(\mathbf{A}+\Box\theta) = \mathbf{F} - M(\mathbf{A}+\Box\theta) \tag{13.21}$$

Now apply $\Box - M$ again and put $\Box \bullet \mathbf{F} = -\mu_0 \mathbf{J}$

$$\begin{aligned}(\Box - M)\left[\mathbf{F} - M(\mathbf{A} + \Box\theta)\right] &= \Box\mathbf{F} - M\Box\mathbf{A} - M\Box^2\theta - M\mathbf{F} + M^2\mathbf{A} + M\Box\theta \\ &= \Box\mathbf{F} + M^2\mathbf{A} - 2M\mathbf{F} + M\Box\theta \qquad (13.22)\end{aligned}$$

since $\Box\mathbf{A} = \mathbf{F}$ and we take $\Box\theta = 0$, then $\Box^2\theta = 0$. The term $2M\mathbf{F} = \text{Biv}$, the EM field bivector, adds nothing new, therefore we drop it.

$$\begin{aligned}\Box\mathbf{F} &= \mathbf{e}_0\left(-\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z}\right) + \mathbf{e}_1\left[\frac{\partial E_x}{c\partial t} - \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\right] \\ &\quad + \mathbf{e}_2\left[\frac{\partial E_x}{c\partial t} - \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\right] + \mathbf{e}_3\left[\frac{\partial E_z}{c\partial t} - \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\right] \\ &\quad + \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\left[-\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_y}{\partial y}\right) - \frac{\partial B_z}{\partial t}\right] + \mathbf{e}_0\mathbf{e}_3\mathbf{e}_1\left[-\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) - \frac{\partial B_y}{\partial t}\right] \\ &\quad + \mathbf{e}_0\mathbf{e}_2\mathbf{e}_3\left[-\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \frac{\partial B_x}{\partial t}\right] + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) \\ &= \Box \bullet \mathbf{F} + \Box \wedge \mathbf{F} \qquad (13.23)\end{aligned}$$

As in **E** and **M**

$$\Box \bullet \mathbf{F} = -\mu_0\mathbf{J} = -\mu_0 c\rho\mathbf{e}_0 - \mu_0 J_x - \mu_0 J_y - \mu_0 J_z \qquad (13.24)$$

Putting the right hand side of Eq. (13.22) equal to $-\mu_0\mathbf{J}$ and equating coefficients of the unit vectors, we obtain

$$\begin{aligned}-\text{div}\,\mathbf{E} - M^2 A_0 &= -\mu_0 c\rho \\ -(\text{curl}\,\mathbf{B})_x + \frac{\partial E_x}{c\partial t} + M^2 A_1 &= -\mu_0 j_x \\ -(\text{curl}\,\mathbf{B})_y + \frac{\partial E_y}{c\partial t} + M^2 A_2 &= -\mu_0 j_y \\ -(\text{curl}\,\mathbf{B})_z + \frac{\partial E_z}{c\partial t} + M^2 A_3 &= -\mu_0 j_z \\ \text{or}\quad \text{div}\,E - M^2 A_0 &= \frac{\rho}{\varepsilon_0} \\ (\text{curl}\,\mathbf{B})_x - \frac{\partial E_x}{c\partial t} - M^2 A_1 &= \mu_0 j_x \\ (\text{curl}\,\mathbf{B})_y - \frac{\partial E_y}{c\partial t} - M^2 A_2 &= \mu_0 j_y \\ (\text{curl}\,\mathbf{B})_z - \frac{\partial E_z}{c\partial t} - M^2 A_3 &= \mu_0 j_z \qquad (13.25)\end{aligned}$$

or

$$\begin{aligned} \operatorname{div}\mathbf{E} - M^2 A_0 &= \frac{\rho}{\varepsilon_0} \\ \operatorname{curl}\mathbf{B} - M^2 \mathbf{A} &= \mu_0 \mathbf{j} \end{aligned} \tag{13.26}$$

The $\square \wedge F = 0$ equations are the same as E and M, namely

$$\begin{aligned} \operatorname{div}\mathbf{B} &= 0 \\ \operatorname{curl}\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \end{aligned} \tag{13.27}$$

Eqs. (13.26) and (13.27) are the Proca field equations.

In Eqs. (13.26) and (13.27) we express the field quantities **E** and **B** in terms of their definitions

$$\begin{aligned} \frac{E_x}{c} &= -\frac{\partial A_0}{\partial x} - \frac{\partial A_1}{c\partial t} \qquad B_x = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \\ \frac{E_y}{c} &= -\frac{\partial A_0}{\partial y} - \frac{\partial A_2}{c\partial t} \qquad B_y = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \\ \frac{E_z}{c} &= -\frac{\partial A_0}{\partial z} - \frac{\partial A_3}{c\partial t} \qquad B_z = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{aligned}$$

to obtain

$$\begin{aligned} & \frac{\partial}{\partial x}\left(-\frac{\partial A_0}{\partial x} - \frac{\partial A_1}{\partial t}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial A_0}{\partial y} - \frac{\partial A_2}{\partial t}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial A_0}{\partial z} - \frac{\partial A_3}{\partial t}\right) \\ + \; & \frac{\partial^2 A_0}{c^2\partial t^2} - \frac{\partial^2 A_0}{c^2\partial t^2} - M^2 A_0 = \frac{\rho}{\varepsilon_0} \end{aligned}$$

$$\begin{aligned} & \frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) - \frac{\partial}{c\partial t}\left(-\frac{\partial A_0}{\partial x} - \frac{\partial A_1}{c\partial t}\right) \\ + \; & \frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial x^2} - M^2 A_1 = \mu_0 j_x \end{aligned}$$

$$\begin{aligned} & \frac{\partial}{\partial z}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{c\partial t}\left(-\frac{\partial A_0}{\partial y} - \frac{\partial A_2}{\partial t}\right) \\ + \; & \frac{\partial^2 A_2}{\partial y^2} - \frac{\partial^2 A_2}{\partial y^2} - M^2 A_2 = \mu_0 j_y \end{aligned}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_1}{\partial z}\right)-\frac{\partial}{\partial y}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)-\frac{\partial}{c\partial t}\left(-\frac{\partial A_0}{\partial z}-\frac{\partial A_0}{c\partial t}\right)$$
$$+\quad \frac{\partial^2 A_3}{\partial z^2}-\frac{\partial^2 A_3}{\partial z^2}-M^2A_3=\mu_0 j_z$$

where we have added and subtracted a term on each line to obtain

$$\begin{aligned}
-\left[\frac{\partial^2 A_0}{\partial x^2}+\frac{\partial^2 A_0}{\partial y^2}+\frac{\partial^2 A_0}{\partial z^2}+\frac{\partial^2 A_0}{c^2\partial t^2}\right]-\frac{\partial}{\partial t}\left[\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}+\frac{\partial A_0}{\partial t}\right]-M^2A_0 &= \frac{\rho}{\varepsilon_0}\\
-\left[\frac{\partial^2 A_1}{\partial x^2}+\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}+\frac{\partial^2 A_1}{c^2\partial t^2}\right]-\frac{\partial}{\partial x}\left[\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}+\frac{\partial A_0}{\partial t}\right]-M^2A_1 &= \mu_0 j_x\\
-\left[\frac{\partial^2 A_2}{\partial x^2}+\frac{\partial^2 A_2}{\partial y^2}+\frac{\partial^2 A_2}{\partial z^2}+\frac{\partial^2 A_2}{c^2\partial t^2}\right]-\frac{\partial}{\partial y}\left[\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}+\frac{\partial A_0}{\partial t}\right]-M^2A_2 &= \mu_0 j_y\\
-\left[\frac{\partial^2 A_3}{\partial x^2}+\frac{\partial^2 A_3}{\partial y^2}+\frac{\partial^2 A_3}{\partial z^2}+\frac{\partial^2 A_3}{c^2\partial t^2}\right]-\frac{\partial}{\partial z}\left[\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}+\frac{\partial A_0}{\partial t}\right]-M^2A_3 &= \mu_0 j_z
\end{aligned}$$

Putting $\frac{\partial A_0}{\partial t}+\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}=\Box\bullet A=0$, we obtain

$$\begin{aligned}
\Box^2 A_0+M^2A_0 &= -\frac{\rho}{\varepsilon_0}\\
\Box^2 A_1+M^2A_1 &= -\mu_0 j_x\\
\Box^2 A_2+M^2A_2 &= -\mu_0 j_y\\
\Box^2 A_3+M^2A_3 &= -\mu_0 j_z
\end{aligned}$$

If the time dependence in $\Box^2$ is ignored so $\Box^2\to\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ and replace ρ by q, the equations become

$$\begin{aligned}
\nabla^2 A_0+M^2A_0 &= -\frac{q}{\varepsilon_0}\\
\nabla^2 A_1+M^2A_1 &= -\mu_0 j_x\\
\nabla^2 A_2+M^2A_2 &= -\mu_0 j_y\\
\nabla^2 A_3+M^2A_3 &= -\mu_0 j_z
\end{aligned}$$

To obtain a decreasing function in r for A_0 we need to replace M^2 by $-M^2$. To achieve this, replace M at the beginning by iM or $\mathbf{e}_5M$. We choose the latter. The equation for A_0 which we now call Φ becomes

$$\nabla^2\phi-M^2\Phi=q$$

The spherically symmetric solution (Yukawa form) is

$$\phi(r) = -\frac{q\mathbf{e}^{-M/r}}{r}.$$

All of the force interaction equations that follow from the Coulomb potential used to describe gravity will apply to Yukawa potential forces by differentiating the Yukawa potential respect to r and using the result along with the form of the velocity interactions that appear in the modified electromagnetic interactions that describe gravitational forces.