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FIELD PRODUCED BY A CONSTANT VELOCITY CHARGE. FIELD MEASURED FROM THE PRESENT POSITION

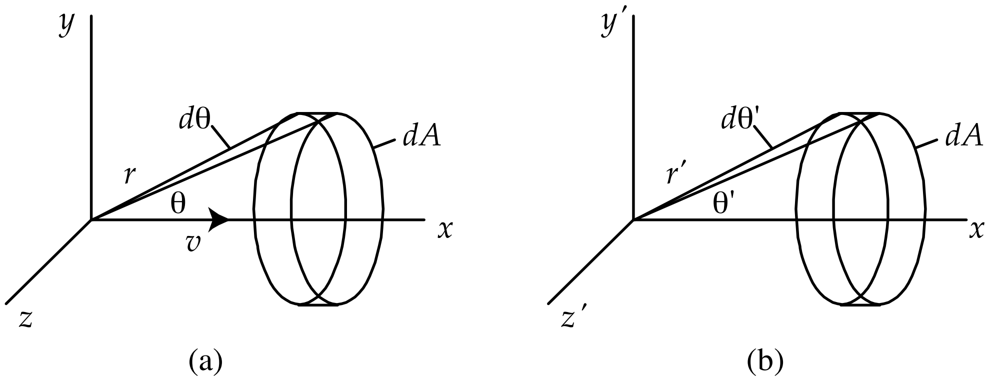

Fig. 12.1

Coordinate axes S' are moving to the right (for example) with velocity $\mathtt{w}$ and carrying charge q at the origin. The axes S and S' are coincident but we show them separated for clarity in labeling.

If we regard S'as moving, then putting S coincident with S' is the same as placing S at the present position of the moving charge q. The electromagnetic field is then described from the present position of the charge rather than from the retarded position of the charge. We will transform the coordinates of the charge as measured in the moving coordinate system to a coordinate system whose origin is coincident with the moving charge.

If we imagine a photon leaving an origin O arriving later at some point P in the field, then the present position of the origin is the position the origin would reach by traveling in a straight line at velocity $\mathtt{w}$ from the point of emission during the time the photon is traveling from the point of emission to the point P. It is also called the simultaneous position. The positions of charges q_s and q_t used in writing the Faraday Law are present positions. Therefore, to be consistent, the charges responsible for the electric field $\mathbf{E}$, must be referred to the same position.

Geodesic flux is defined as the number of geodesics emanating from a source point in a solid angle multiplied by the strength of the source. It is a number and simply involves counting and is therefore a relativistic invariant. The solid angle cone encompassing the geodesics may vary in size from coordinate system to coordinate system as they move at constant velocity with respect to each other. However, the number of flux lines, that is, the number of geodesics in a given cone, remains unchanged.

Fig. (12.1a) shows the field lines of a charge q in a coordinate system in which q is at rest. Now imagine the charge attached to the origin O' of a coordinate system moving with a velocity $\mathtt{w}$ to the right with respect to O. As q passes O the symmetric moving coordinates of the field point measured by O' are given by the primed quantities as labeled in Fig. (12.1b). An observer at rest would see a symmetric field as shown in Fig. (12.1a).

The Lorentz transformations connecting coordinates as measured by S and S' when their origins coincide are:

$$\begin{aligned}
&x' = \gamma\left(x - \mathtt{w}t\right) \rightarrow \qquad x' = \gamma x \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{\mathtt{w}}{c} \\
&y' = y, \qquad z' = z \\
&t' = \gamma\left(t - \frac{\mathtt{w}x}{c^2}\right) \rightarrow \qquad t' = -\gamma\frac{\mathtt{w}x}{c^2}
\end{aligned} \tag{12.1}$$

Two observers with relative velocity $\mathtt{w}$ both observe flux passing through a ring of area circling the x-axis. The area is dS to one and dS' to the other. The Lorentz transformations relate the coordinates of an event at P as measured by S to the coordinates of the same event as measured by S'. The origins are momentarily coincident as q, fixed in S', moves at velocity $\mathtt{w}$ in the x-direction. The element of momentum flux density, df, emanating from q is

$$df = q^2\varepsilon_0\left(\mathbf{E}\times\mathbf{B}\right)\frac{dA_\perp}{r^2}$$

$dA_\perp$ = element of area perpendicular to r.

$$\frac{df}{dA_\perp} = q^2\varepsilon_0\frac{(\mathbf{E}\times\mathbf{B})}{r^2} \tag{12.2}$$

Consider the flux linking a ring of area dA. Denote the projection of that area perpendicular to r by $dA_\perp$. Then according to S:

$$\begin{aligned} df &= \varepsilon_0 (\mathbf{E} \times \mathbf{B})\, r d\theta 2\pi r \sin\theta d\theta \\ &= \varepsilon_0 (\mathbf{E} \times \mathbf{B})\, 2\pi r^2 \sin\theta d\theta \end{aligned} \tag{12.3}$$

The element of flux is invariant but the area of the ring as measured by S' is $dA'_\perp$, thus according to S':

$$df' = \varepsilon_0 (\mathbf{E}' \times \mathbf{B}')\, dA'_\perp = \varepsilon_0 (\mathbf{E}' \times \mathbf{B}')\, 2\pi r'^2 \sin\theta' d\theta' \tag{12.4}$$

Equating Eqs. (12.3) and (12.4)

$$\frac{\varepsilon_0 (\mathbf{E} \times \mathbf{B})}{\varepsilon_0 (\mathbf{E}' \times \mathbf{B}')} = \frac{2\pi r^2 \sin\theta d\theta}{2\pi r'^2 \sin\theta' d\theta'}$$

The motion is in the x-direction, so the y and z coordinates are the same for each observer thus

$$r \sin\theta = r' \sin\theta'$$

Therefore

$$\frac{\varepsilon_0 (\mathbf{E} \times \mathbf{B})}{\varepsilon_0 (\mathbf{E}' \times \mathbf{B}')} = \frac{r d\theta}{r' d\theta'} \tag{12.5}$$

The Lorentz transformation of r and $\sin\theta$ in polar coordinates is obtained as follows:

Observer S' sitting at the origin of his coordinate system (and in this case carrying a charge q) passes S with a velocity $\mathtt{w}$ in the x-direction. At the moment the axes coincide the coordinates of a point x at $t = 0$ as measured by S are related to the same point and time (space-time event) as measured by S' by the Lorentz transformations, Eq. (12.1). Thus

$$\begin{aligned} r'^2 &= x'^2 + y'^2 + z'^2 = \gamma^2 x^2 + y^2 + z^2 = \gamma^2 \left(x^2 + \frac{y^2 + z^2}{\gamma^2} \right) \\ r'^2 &= \gamma^2 \left[x^2 + y^2 + z^2 - \beta^2 \left(y^2 + z^2 \right) \right] \\ &= \gamma^2 \left[r^2 - \beta^2 \left(y^2 + z^2 \right) \right] = \gamma^2 r^2 \left[1 - \beta^2 \frac{(y^2 + z^2)}{r^2} \right] \end{aligned}$$

$$\text{But} \qquad \frac{y^2+z^2}{r^2} = \sin^2\theta$$

$$\text{Therefore} \qquad r' = \gamma r\left(1-\beta^2\sin^2\theta\right)^{1/2} \tag{12.6}$$

$$\text{and} \qquad \frac{1}{r'^2} = \frac{1}{\gamma^2 r^2\left(1-\beta^2\sin^2\theta\right)}$$

$$\text{Now use} \qquad r'\sin\theta' = r\sin\theta$$

$$\sin\theta' = \frac{r}{r'}\sin\theta$$

$$\text{Therefore} \qquad \sin\theta' = \frac{\sin\theta}{\gamma\left(1-\beta^2\sin^2\theta\right)^{1/2}} \tag{12.7}$$

To relate $d\theta$ and $d\theta'$ write the differential of Eq. (12.7) to obtain

$$\cos\theta' d\theta' = \left[\frac{\left(1-\beta^2\sin^2\theta\right)}{\gamma\left(1-\beta^2\sin^2\theta\right)^{3/2}}\cos\theta + \sin\theta\frac{\beta^2\sin\theta\cos\theta}{\gamma\left(1-\beta^2\sin^2\theta\right)^{3/2}}\right]d\theta$$

$$\cos\theta' d\theta' = \frac{\cos\theta d\theta}{\gamma\left(1-\beta^2\sin^2\theta\right)^{3/2}}$$

$$\frac{d\theta}{d\theta'} = \frac{\cos\theta'\gamma\left(1-\beta^2\sin^2\theta\right)^{3/2}}{\cos\theta} \tag{12.8}$$

From Eq. (12.7)

$$\cos^2\theta' = 1-\sin^2\theta' = 1-\frac{\sin^2\theta}{\gamma^2\left(1-\beta^2\sin^2\theta\right)}$$

$$= \frac{\gamma^2\left(1-\beta^2\sin^2\theta-\sin^2\theta/\gamma^2\right)}{\gamma^2\left(1-\beta^2\sin^2\theta\right)} \qquad \beta^2+\frac{1}{\gamma^2} = \beta^2+1-\beta^2 = 1$$

$$= \frac{1-\sin^2\theta}{1-\beta^2\sin^2\theta}$$

$$\cos^2\theta' = \frac{\cos^2\theta}{1-\beta^2\sin^2\theta}$$

$$\cos\theta' = \frac{\cos\theta}{\left(1-\beta^2\sin^2\theta\right)^{1/2}} \tag{12.9}$$

Put this in Eq. (12.8)

$$\begin{aligned} d\theta' &= \frac{\cos\theta d\theta}{\cos\theta'\gamma\left(1-\beta^2\sin^2\theta\right)^{3/2}} = \frac{\cos\theta\left(1-\beta^2\sin^2\theta\right)^{1/2}d\theta}{\gamma\cos\theta\left(1-\beta^2\sin^2\theta\right)^{3/2}} \\ d\theta' &= \frac{d\theta}{\gamma\left(1-\beta^2\sin^2\theta\right)} \qquad (12.10) \end{aligned}$$

By flux conservation:

$$\begin{aligned} \varepsilon_0\left(\mathbf{E}'\times\mathbf{B}'\right)r'd\theta' &= \varepsilon_0\left(\mathbf{E}\times\mathbf{B}\right)rd\theta \\ \frac{\varepsilon_0\left(\mathbf{E}'\times\mathbf{B}'\right)}{\varepsilon_0\left(\mathbf{E}\times\mathbf{B}\right)} &= \frac{rd\theta}{r'd\theta'} = \frac{\gamma\left(1-\beta^2\sin^2\theta\right)}{\gamma\left(1-\beta^2\sin^2\theta\right)^{1/2}} \qquad (12.11) \\ &= \left(1-\beta^2\sin^2\theta\right)^{1/2} \\ (EB) &= \frac{(E'B')}{\left(1-\beta^2\sin^2\theta\right)^{1/2}} = \frac{q^2}{4\pi\varepsilon_0 r'^2}\frac{1}{\left(1-\beta^2\sin^2\theta\right)^{1/2}} \qquad (12.12) \\ (EB) &= \frac{q^2}{4\pi\varepsilon_0 r^2}\frac{1-\beta^2}{\left(1-\beta^2\sin^2\theta\right)^{3/2}} \qquad (12.13) \end{aligned}$$

In the following we expand EB as given by Eq. (12.13) up to order $1/c^2$.

$$EB = \frac{q_s^2}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}\frac{(1-\beta^2)}{(1-\beta^2\sin^2\theta)^{3/2}}$$

$$\begin{aligned} (1-x)^{-n} &= 1+nx+\frac{n(n+1)x^2}{2!} = 1+\frac{3}{2}x+\frac{15}{8}x^2 \\ \text{For } n &= 3/2 \text{ and } x = \beta^2\sin^2\theta \end{aligned}$$

$$\begin{aligned} \frac{1}{(1-\beta^2\sin^2\theta)^{3/2}} &= 1+\frac{3}{2}\beta^2\sin^2\theta+\frac{15}{8}\beta^4\sin^4\theta \\ &= 1+\frac{3}{2}\beta^2-\frac{3}{2}\beta^2\cos^2\theta+\frac{15}{8}\beta^4\left(1-2\cos^2\theta+\cos^4\theta\right) \\ \frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}} &= 1+\frac{3}{2}\beta^2-\frac{3}{2}\beta^2\cos^2\theta-\beta^2 \\ &= 1+\frac{\beta^2}{2}-\frac{3\beta^2}{2}\cos^2\theta \end{aligned}$$

Therefore, up to v^2/c^2

$$\varepsilon_0\left(\mathbf{E}\times\mathbf{B}\right)=\frac{q^2}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2}+\frac{q^2}{4\pi\varepsilon_0}\frac{1}{2c^2}\left[\frac{v^2\hat{\mathbf{r}}}{r^2}-\frac{3(\mathbf{v}\cdot\hat{\mathbf{r}})^2\hat{\mathbf{r}}}{r^2}\right] \tag{12.14}$$

or when acceleration is included

$$\varepsilon_0\left(\mathbf{E}\times\mathbf{B}\right)=\frac{q^2}{4\pi\varepsilon_0}\left[\frac{\mathbf{r}}{r^3}+\frac{1}{2c^2}\left[\frac{v^2\mathbf{r}}{r^3}-\frac{3(\mathbf{v}\cdot\mathbf{r})^2\mathbf{r}}{r^5}-\frac{\mathbf{a}}{r}-\frac{(\mathbf{a}\cdot\mathbf{r})\,\mathbf{r}}{r^3}\right]\right] \tag{12.15}$$

E is in the direction of the radius vector $\hat{\mathbf{r}}$. One could denote E' by E_0 since it refers to E in the S' rest frame. Likewise θ' by θ_0. Denoting the components of $\mathbf{E}$ parallel and normal to the velocity by subscripts p and n respectively. In terms of E', using Eqs. (12.7) and (12.9).

$$E_p = \frac{E'\cos\theta}{\left(1-\beta^2\sin^2\theta\right)^{1/2}} = E'\cos\theta' = E'_p \tag{12.16}$$

$$E_n = \frac{E'\sin\theta}{\left(1-\beta^2\sin^2\theta\right)^{1/2}} = \frac{E'\sin\theta'}{\sqrt{1-\beta^2}} = \frac{E'_n}{\sqrt{1-\beta^2}} \tag{12.17}$$

To obtain the magnitude of the corresponding magnetic field, use:

$$\text{curl}\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{\partial\mathbf{E}}{c\partial t} \tag{12.18}$$

Integrate Eq. (12.18) over the surface of the spherical cap.

$$\int(\text{curl}\mathbf{B})\cdot d\mathbf{A} = \frac{1}{c}\int\frac{\partial\mathbf{E}}{\partial t}\cdot d\mathbf{A} \tag{12.19}$$

By Stokes' theorem:

$$\int\text{curl}\mathbf{B}\cdot d\mathbf{A} = \int\mathbf{B}\cdot d\mathbf{l} = B2\pi r\sin\theta$$

$$d\mathbf{l} = \text{element of length of the curve around the cap}$$

$$B2\pi r\sin\theta = \frac{1}{c}\frac{\partial}{\partial t}\int_0^\theta Erd\theta 2\pi r = \frac{1}{c}\frac{\partial}{\partial t}\int_0^\theta E2\pi r^2\sin\theta d\theta$$

$$B=\beta E_n = E\beta\sin\theta = \frac{q_s\left(1-\beta^2\right)\beta\sin\theta}{4\pi r^2\left(1-\beta^2\sin^2\theta\right)^{3/2}} = \frac{\beta E'_n}{\sqrt{1-\beta^2}} \tag{12.20}$$

Since E', E'_n and E'_p are measured at rest with respect to the coordinate system S', they could, for clarity, also be labeled with a zero subscript such as E'_0, E'_{0n} and E'_{0p}, E'_{0x}, etc. to clearly show that they are measured in the rest coordinates of S'. The same applies to r', θ' which could be labeled r'_0, θ'_0 for clarity.