10

CALCULATION OF DIRECT AND INDIRECT FORCES AND TORQUES

10.1 Action and Reaction Between Two Moving Charges. Gravitational-Orbital Motion

The following continues with the progress outlined in Chapter 9, but now shows details for modifying standard electromagnetism results to obtain gravitation forces between two interacting masses. Electromagnetism and gravity obey the same equations. However, electric charges are introduced when describing electromagnetism and mass is introduced when describing gravity.

We label one charge q_s for source charge and the other q_t for test charge. The roles are interchangeable.

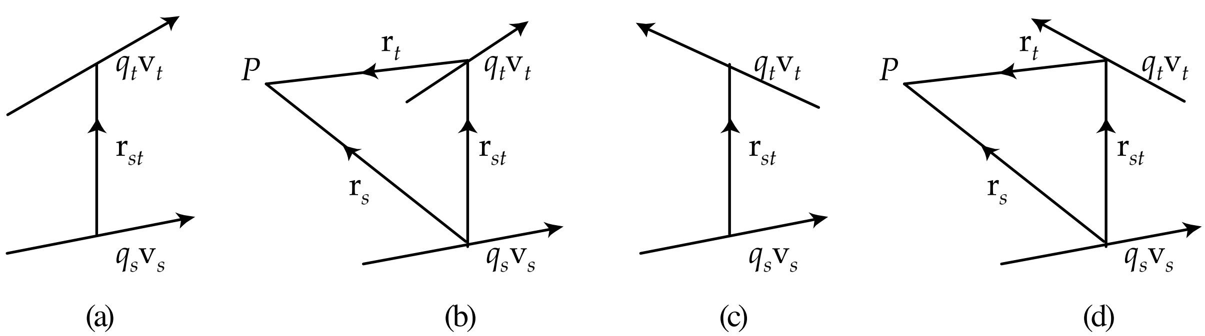

Fig. 10.1

10.2 Verification that Action and Reaction are Not Equal and Opposite in the Faraday Law

By the Biot-Savart law, the magnetic force field at the position of $q_t\mathbf{v}_t$ generated by the motion of $q_s\mathbf{v}_s$, Fig. 10.1a, is

$$\mathbf{B}_s = \frac{\mu_0 q_s\left(\mathbf{v}_s \times \mathbf{r}_{st}\right)}{4\pi r^3}$$

All quantities will be in SI units. $r^3 = |r_{st}|^3 = |r_{ts}|^3$. Relativistic velocities may be introduced by replacing $\mathbf{v}_s$ by $\gamma_s\mathbf{v}_s$ and $\mathbf{v}_t$ by $\gamma_t\mathbf{v}_t$, where $\gamma_s = 1/\left(1 - v_s^2/c^2\right)^{1/2}$ and $\gamma_t = 1/\left(1 - v_t^2/c^2\right)^{1/2}$. For the present, we are interested in displaying an accuracy up to v^2/c^2 so that only the electromagnetic field $\mathbf{E}$ need be relativistic up to v^2/c^2. The magnetic force is valid to v^2/c^2 *ab initio*.

The magnetic field generated by $q_t\mathbf{v}_t$ at $q_s\mathbf{v}_s$ is

$$\mathbf{B}_t = \frac{\mu_0 q_t\left(\mathbf{v}_t \times \mathbf{r}_{ts}\right)}{4\pi r^3} = \frac{q_t q_s}{c^2} \times \frac{\mathbf{r}_{st}}{4\pi\varepsilon_0 c^2 r^3} = \frac{q_t\mathbf{v}_t}{c^2} \times \mathbf{E}_t$$

10.3 1. Magnetic Force

In the following we write the product $q_t q_s$ as $\left(q_t q_s + q_s q_t\right)/2$ to emphasize that the results apply to all space-time multivectors q_s and q_t of like kind (Page and Adams 1945). This switch could be delayed until the end but it is introduced initially to gain familiarity with the idea. Results apply to electromagnetism in the usual formalism by simply replacing $\left(q_t q_s + q_s q_t\right)/2$ by $q_t q_s$. For explicit inclusion in Eqs. (10.1) and (10.2), replace $q_t q_s$ by $\left(q_t q_s + q_s q_t\right)/2$ and replace $q_s q_t$ by$\left(q_s q_t + q_t q_s\right)/2$.

By Faraday's Law, the direct force on $q_t\mathbf{v}_t$ from the field generated by $q_s\mathbf{v}_s$ using $\mu_0\varepsilon_0 = 1/c^2$ is

$$\begin{aligned}\mathbf{F}_{ts}^{\mathbf{B}_s} &= \frac{\mu_0\left[\mathbf{v}_t \times \left(\mathbf{v}_s \times \mathbf{r}_{st}\right)\right]}{4\pi r_{st}^3}\frac{\left(q_s q_t + q_t q_s\right)}{2} \\ &= \frac{\left[\mathbf{v}_s\left(\mathbf{v}_t \cdot \mathbf{r}_{st}\right) - \left(\mathbf{v}_t \cdot \mathbf{v}_s\right)\mathbf{r}_{st}\right]}{4\pi\varepsilon_0 c^2 r^3}\frac{\left(q_s q_t + q_t q_s\right)}{2} \qquad (10.1)\end{aligned}$$

attractive for Fig. (10.1c)

where $\mathbf{F}_{ts}^{\mathbf{B}_s}$ stands for the direct magnetic force, also called the Faraday force, of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$. We use the word "direct" to denote a point to point interaction. No integration over space is involved.

The direct magnetic force of $q_t\mathbf{v}_t$ on $q_s\mathbf{v}_s$ is

$$\begin{aligned}\mathbf{F}_{st}^{\mathbf{B}_t} &= \frac{\mu_0\mathbf{v}_s\times(\mathbf{v}_t\times\mathbf{r}_{ts})}{4\pi r_{ts}^3}\frac{(q_tq_s+q_sq_t)}{2} = \frac{[\mathbf{v}_t(\mathbf{v}_s\cdot\mathbf{r}_{ts})-(\mathbf{v}_s\cdot\mathbf{v}_t)\mathbf{r}_{ts}]}{4\pi\varepsilon_0c^2r^3}\frac{(q_tq_s+q_sq_t)}{2}\\ &= \frac{[-\mathbf{v}_t(\mathbf{v}_s\cdot\mathbf{r}_{st})+(\mathbf{v}_s\cdot\mathbf{v}_t)\mathbf{r}_{st}]}{4\pi\varepsilon_0c^2r^3}\frac{(q_tq_s+q_sq_t)}{2} \qquad (10.2)\end{aligned}$$

Clearly $(q_sq_t+q_tq_s)=(q_tq_s+q_sq_t)$. Also $\mathbf{r}_{st}=-\mathbf{r}_{ts}$. Adding Eqs. (10.1) and (10.2)

$$\begin{aligned}\mathbf{F}_{ts}^{\mathbf{B}_s}+\mathbf{F}_{st}^{\mathbf{B}_t} &= \frac{[\mathbf{v}_s(\mathbf{v}_t\cdot\mathbf{r}_{st})-\mathbf{v}_t(\mathbf{v}_s\cdot\mathbf{r}_{st})]}{4\pi\varepsilon_0c^2}\frac{(q_sq_t+q_tq_s)}{2}\\ &= \frac{[\mathbf{r}_{st}\times(\mathbf{v}_s\times\mathbf{v}_t)]}{4\pi\varepsilon_0c^2r^3}\frac{(q_sq_t+q_tq_s)}{2}\\ &= \frac{(\mathbf{v}_t\times\mathbf{v}_s)\times\mathbf{r}_{st}}{4\pi\varepsilon_0c^2r^3}\frac{(q_sq_t+q_tq_s)}{2} \qquad (10.3)\end{aligned}$$

In the above $\mathbf{F}_{ts}^{\mathbf{B}}+\mathbf{F}_{st}^{\mathbf{B}}$ is zero only when (a) $\mathbf{v}_t$ is parallel to $\mathbf{v}_s$ or (b) $\mathbf{v}_t$ and $\mathbf{v}_s$ are both perpendicular to $\mathbf{r}_{st}$. $\mathbf{F}_{ts}^{\mathbf{B}}\neq-\mathbf{F}_{st}^{\mathbf{B}}$ but we have not included the direct electric force, that is, the Coulomb force of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$ with velocity and acceleration corrections; nor have we included the "indirect" electromagnetic force of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$ associated with the reaction force generated by the time rate of change of the linear momentum delivered to the field jointly by $q_s\mathbf{v}_s$ and $q_t\mathbf{v}_t$ given by the time rate of change of the Poynting vector for the linear momentum.

The two direct forces to a first approximation are the Coulomb force and the Faraday force. The distances are measured from the present position of the source, that is, the position of the source at the time the force is experienced by $q_t\mathbf{v}_t$. These are also called the simultaneous positions.

10.4 2. Coulomb Force

The direct Coulomb force ($q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$) to order v^2/c^2 is given by

$$\mathbf{F}_{ts}^{\mathbf{E}_s}= \frac{1}{4\pi\varepsilon_0c^2}\left[\frac{c^2\mathbf{r}_{st}}{r^3}+\frac{1}{2}\left(\frac{v_s^2\mathbf{r}_{st}}{r^3}-\frac{3(\mathbf{v}_s\cdot\mathbf{r}_{st})^2\mathbf{r}_{st}}{r^5}-\frac{\mathbf{a}_s}{r}-\frac{(\mathbf{a}_s\cdot\mathbf{r}_{st})\mathbf{r}_{st}}{r^3}\right)\right]\frac{(q_tq_s+q_sq_t)}{2} \qquad (10.4)$$

The Coulomb force includes a velocity and acceleration correction to order $1/c^2$.

The corresponding direct Coulomb force of $q_t\mathbf{v}_t$ on $q_s\mathbf{v}_s$ is obtained by interchanging s and t in Eq. (10.4). We also replace $\mathbf{r}_{ts}$ by $-\mathbf{r}_{st}$ in order to more readily compare the two sets. Thus the Coulomb force ($q_t\mathbf{v}_t$ on $q_s\mathbf{v}_s$) is given by

$$\mathbf{F}_{st}^{\mathbf{E}_t} = \frac{1}{4\pi\varepsilon_0 c^2}\left[-\frac{c^2\mathbf{r}_{st}}{r^3}+\frac{1}{2}\left(-\frac{v_t^2\mathbf{r}_{st}}{r^3}+\frac{3\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5}-\frac{\mathbf{a}_t}{r}-\frac{\left(\mathbf{a}_t\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\right)\right]\frac{\left(q_tq_s+q_sq_t\right)}{2} \tag{10.5}$$

Note that $\mathbf{F}_{ts}^{\mathbf{E}_s}$ only satisfies the third law when $\mathbf{a}_s = -\mathbf{a}_t$ and $\mathbf{v}_s = \pm\mathbf{v}_t$ only then does $\mathbf{F}_{st}^{\mathbf{E}_t} = -\mathbf{F}_{ts}^{\mathbf{E}_s}$. Therefore in general

$$\mathbf{F}_{ts}^{\mathbf{E}_s} + \mathbf{F}_{st}^{\mathbf{E}_t} \neq 0$$

Thus, in general, the sum

$$\left(\mathbf{F}_{ts}^{\mathbf{B}_s} + \mathbf{F}_{ts}^{\mathbf{E}_s}\right) + \left(\mathbf{F}_{st}^{\mathbf{B}_t} + \mathbf{F}_{st}^{\mathbf{E}_t}\right) \neq 0$$

and the third law does not hold for the sum of the above "direct" forces. In the above equations, the forces are described from the present position of the particles.

10.5 3. Indirect Force (Electromagnetic Force)

To obtain the correct total force acting on $q_t\mathbf{v}_t$ caused by $q_s\mathbf{v}_s$ and vice versa, one must include the indirect force, also called the electromagnetic force, experienced by both current elements via the time rate of change of the linear momentum that they jointly deliver to the field as expressed by the Poynting momentum vector. We label the force experienced by $q_t\mathbf{v}_t$ as a result of this process $\mathbf{F}_{ts}$. We retain the second subscript since the force is generated by the mutual interaction of $q_s\mathbf{v}_s$ and $q_t\mathbf{v}_t$ and not exclusively by $q_s\mathbf{v}_s$. Likewise we denote the indirect force on $q_s\mathbf{v}_s$ by $\mathbf{F}_{st}$. The specific terms that describe the force will be identified later by a superscript on $\mathbf{F}_{ts}$ and on $\mathbf{F}_{st}$. The total linear momentum density at a point in the field is given by the Poynting momentum vector

$$\begin{aligned}\varepsilon_0\left(\mathbf{E}\times\mathbf{B}\right) &= \varepsilon_0\left(\mathbf{E}_s+\mathbf{E}_t\right)\times\left(\mathbf{B}_s+\mathbf{B}_t\right)\\ &= \varepsilon_0\left[\mathbf{E}_s\times\mathbf{B}_s+\mathbf{E}_s\times\mathbf{B}_t+\mathbf{E}_t\times\mathbf{B}_s+\mathbf{E}_t\times\mathbf{B}_t\right]\end{aligned}$$

The mutual linear momentum density is

$$\mathbf{g}_\ell = \varepsilon_0\left[\mathbf{E}_s\times\mathbf{B}_t+\mathbf{E}_t\times\mathbf{B}_s\right] \tag{10.6}$$

Thus $q_s\mathbf{v}_s$ and $q_t\mathbf{v}_t$ jointly generate linear momentum density in the field in accordance with Eq. (10.6). The integral of $\mathbf{g}_\ell$ over all space gives the total linear momentum delivered to the field by the mutual interaction of $q_s\mathbf{v}_s$ and $q_t\mathbf{v}_t$. We label this momentum $\mathbf{G}_\ell$,

$$\mathbf{G}_\ell = \varepsilon_0 \int (\mathbf{E}_s \times \mathbf{B}_t)\, d\tau + \varepsilon_0 \int (\mathbf{E}_t \times \mathbf{B}_s)\, d\tau \tag{10.7}$$

The force on the field, and indirectly on the particles themselves is

$$\frac{d\mathbf{G}_\ell}{dt} = \varepsilon_0 \frac{d}{dt} \int (\mathbf{E}_s \times \mathbf{B}_t)\, d\tau + \varepsilon_0 \frac{d}{dt} \int (\mathbf{E}_t \times \mathbf{B}_s)\, d\tau \tag{10.8}$$

In generating the force on the field $q_t\mathbf{v}_t$ and $q_s\mathbf{v}_s$ each will experience a reactive force. We now ask which of the two terms in Eq. (3a) does one associate with a reactive force on $q_s\mathbf{v}_s$ and with which do we associate a reactive force on $q_t\mathbf{v}_t$ or perhaps it is some combination of the two.

We argue that the correct assignment for the total force experienced by $q_t\mathbf{v}_t$ is given by

$$\mathbf{F}_{ts}^{T} = \overset{\text{indir}}{\mathbf{F}_{ts}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau}} + \overset{\text{dir}}{\mathbf{F}_{ts}^{\mathbf{E}_s}} + \overset{\text{dir}}{\mathbf{F}_{ts}^{\mathbf{B}_s}} \tag{10.9}$$

while the total force on $q_s\mathbf{v}_s$ is

$$\mathbf{F}_{st}^{T} = \overset{\text{indir}}{\mathbf{F}_{st}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau}} + \overset{\text{dir}}{\mathbf{F}_{st}^{\mathbf{E}_s}} + \overset{\text{dir}}{\mathbf{F}_{st}^{\mathbf{B}_t}}$$

The indirect force requires integration over all space and differentiation of the result with respect to t while the direct force, the second and third terms in Eq. (10.9), is a point to point interaction and involves no integration. We have identified the indirect force on $q_t\mathbf{v}_t$ as $\overset{\text{indir}}{\mathbf{F}_{ts}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau}}$. It will be shown that the terms describing the indirect force acting on $q_t\mathbf{v}_t$ up to terms in $1/c^2$ are Eqs. (10.38), (10.39), and (10.40).

$$\begin{aligned}
&\overset{\text{indir}}{\mathbf{F}_{ts}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau}} = \frac{d\mathbf{G}_{\ell ts}^{\varepsilon_0\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau}}{dt} = \\
&\frac{1}{4\pi\varepsilon_0 c^2}\left[\frac{\mathbf{a}_t}{2r} + \frac{(\mathbf{a}_t\cdot\mathbf{r}_{st})\,\mathbf{r}_{st}}{2r^3}\right. \\
&+\frac{\mathbf{v}_t\,(\mathbf{v}_s\cdot\mathbf{r}_{st})}{2r^3} - \frac{(\mathbf{v}_t\cdot\mathbf{v}_s)\,\mathbf{r}_{st}}{2r^3} + \frac{v_t^2\mathbf{r}_{st}}{2r^3} - \frac{(\mathbf{v}_t\cdot\mathbf{r}_{st})\,\mathbf{v}_s}{2r^3} \\
&\left.+\frac{3\,(\mathbf{v}_t\cdot\mathbf{r}_{st})\,(\mathbf{v}_s\cdot\mathbf{r}_{st})\,\mathbf{r}_{st}}{2r^5} - \frac{3\,(\mathbf{v}_t\cdot\mathbf{r}_{st})^2\,\mathbf{r}_{st}}{2r^5}\right](q_tq_s + q_sq_t)/2
\end{aligned} \tag{10.10}$$

We retain the subscript notation ts although the force on $q_t\mathbf{v}_t$ is not due solely to $q_s\mathbf{v}_s$. We identify it as an indirect force on $q_t\mathbf{v}_t$ as a result of $q_s\mathbf{v}_s$ and $q_t\mathbf{v}_t$ jointly generating momentum in the field. The reaction force on $q_t\mathbf{v}_t$ is labeled as the force of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$.

The group of terms Eq. (10.10) is obtained as follows.

The total linear momentum generated by the term $\varepsilon_0 (\mathbf{E}_s \times \mathbf{B}_t)$ is

$$\begin{aligned}\int\limits_V \varepsilon_0 (\mathbf{E}_s \times \mathbf{B}_t)\, d\tau &= \mathbf{G}_\ell^{\varepsilon_0 \int (\mathbf{E}_s \times \mathbf{B}_t) d\tau} = \frac{q_t q_s}{c^2 \mu_0} \int\limits_V \left[\frac{\mathbf{r}_s}{4\pi\varepsilon_0 r_s^3} \times \frac{(\mathbf{v}_t \times \mathbf{r}_s)}{4\pi\varepsilon_0 c^2 r_t^3} \right] d\tau \\ &= \frac{q_t q_s}{(4\pi\varepsilon_0)^2} \frac{\varepsilon_0}{c^2} \int\limits_V \frac{[\mathbf{r}_s \times (\mathbf{v}_t \times \mathbf{r}_s)]}{r_s^3 r_t^3} d\tau \end{aligned} \tag{10.11}$$

After integration over all space, the result is:

$$\mathbf{G}_\ell^{\varepsilon_0 \int (\mathbf{E}_s \times \mathbf{B}_t) d\tau} = \int\limits_V \frac{\varepsilon_0 (\mathbf{E}_s \times \mathbf{B}_t)}{c^2 \mu_0} d\tau = \frac{q_s q_t}{4\pi\varepsilon_0} \left\{ \frac{1}{2c^2} \left[\frac{\mathbf{v}_t}{r} + \frac{(\mathbf{v}_t \cdot \mathbf{r}_{st})\, \mathbf{r}_{st}}{r^3} \right] \right\} \tag{10.12}$$

Details for evaluating the time derivative of Eq. (10.12) are given in Section 10.8. The result is Eq. (10.15) below.

Direct Coulomb force ($q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$),to $1/c^2$

$$\begin{aligned}\mathbf{F}_{ts}^{\mathbf{E}_s} = \frac{1}{4\pi\varepsilon_0} &\left[\frac{\mathbf{r}_{st}}{r^3} + \frac{1}{2c^2} \left(\frac{v_s^2 \mathbf{r}_{st}}{r^3} - \frac{3 (\mathbf{v}_s \cdot \mathbf{r}_{st})^2 \mathbf{r}_{st}}{r^5} \right.\right. \\ &\left.\left. - \frac{\mathbf{a}_s}{r} - \frac{(\mathbf{a}_s \cdot \mathbf{r}_{st})\, \mathbf{r}_{st}}{r^3} \right) \right] (q_t q_s + q_s q_t)/2 \qquad \text{Repulsive electric field}\end{aligned} \tag{10.13}$$

Direct Faraday magnetic force ($q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$)

$$\mathbf{F}_{ts}^{\mathbf{B}_s} = \frac{1}{4\pi\varepsilon_0} \frac{1}{2c^2} \left[\frac{2\mathbf{v}_s (\mathbf{v}_t \cdot \mathbf{r}_{st})}{r^3} - \frac{2 (\mathbf{v}_t \cdot \mathbf{v}_s)\, \mathbf{r}_{st}}{r^3} \right] (q_t q_s + q_s q_t)/2 \qquad \text{Repulsive magnetic field} \tag{10.14}$$

Indirect Electromagnetic field force ($q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$) (repulsive) (Eq. (10.41)

$$\begin{aligned}\mathbf{F}_{ts}^{\varepsilon_0 \frac{d}{dt} \int (\mathbf{E}_s \times \mathbf{B}_t) d\tau} &= \frac{d\mathbf{G}_\ell^{\varepsilon_0 \int (\mathbf{E}_s \times \mathbf{B}_t) d\tau}}{dt} = \frac{1}{4\pi\varepsilon_0} \frac{1}{2c^2} \left[\frac{\mathbf{a}_t}{r} + \frac{\mathbf{v}_t (\mathbf{v}_s \cdot \mathbf{r}_{st})}{r^3} \right. \\ &+ \frac{(\mathbf{a}_t \cdot \mathbf{r}_{st})\, \mathbf{r}_{st}}{r^3} - \frac{(\mathbf{v}_t \cdot \mathbf{v}_s)\, \mathbf{r}_{st}}{r^3} + \frac{v_t^2 \mathbf{r}_{st}}{r^3} - \frac{(\mathbf{v}_t \cdot \mathbf{r}_{st})\, \mathbf{v}_s}{r^3} \\ &\left. + \frac{3 (\mathbf{v}_t \cdot \mathbf{r}_{st}) (\mathbf{v}_s \cdot \mathbf{r}_{st})\, \mathbf{r}_{st}}{r^5} - \frac{3 (\mathbf{v}_t \cdot \mathbf{r}_{st})^2 \mathbf{r}_{st}}{r^5} \right] (q_t q_s + q_s q_t)/2\end{aligned} \tag{10.15}$$

The sum of the forces, Eqs. (10.13), (10.14), and (10.15), is

$$\mathbf{F}_{ts}^{T} = \overset{\text{indir}}{\mathbf{F}_{ts}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau}} + \overset{\text{dir}}{\mathbf{F}_{ts}^{\mathbf{E}_t}} + \overset{\text{dir}}{\mathbf{F}_{ts}^{\mathbf{B}_t}}$$

$$\begin{aligned}\mathbf{F}_{ts}^{T} = \frac{1}{4\pi\varepsilon_0}\Bigg[& \underline{\frac{\mathbf{r}_{st}}{r^3}} + \frac{1}{2c^2}\Bigg(\frac{\mathbf{v}_t\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)}{r^3} + \underset{-\ -\ -\ -}{\frac{2\mathbf{v}_s\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)}{r^3}} - \underset{-\ -\ -\ -}{\frac{2\left(\mathbf{v}_t\cdot\mathbf{v}_s\right)\mathbf{r}_{st}}{r^3}} \\ & -\frac{\left(\mathbf{v}_t\cdot\mathbf{v}_s\right)\mathbf{r}_{st}}{r^3} + \underline{\frac{v_s^2\mathbf{r}_{st}}{r^3}} + \frac{v_t^2\mathbf{r}_{st}}{r^3} + \frac{3\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^5} - \frac{3\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5} \\ & - \underline{\frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5}} + \frac{\mathbf{a}_t}{r} + \frac{\left(\mathbf{a}_t\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3} - \underline{\frac{\mathbf{a}_s}{r}} - \underline{\frac{\left(\mathbf{a}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}}\Bigg)\Bigg]\frac{\left(q_tq_s+q_sq_t\right)}{2}\end{aligned} \quad (10.16)$$

____ Underlined terms come from $\mathbf{F}_{ts}^{\mathbf{E}_s}$
_ _ _ Dashed terms come from $\mathbf{F}_{ts}^{\mathbf{B}_s}$
Remaining terms come from $\mathbf{F}_{ts}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau}$ $\quad$ $\mathbf{a}_t = -v_t^2\mathbf{r}_{st}/r^2$
The corresponding forces on $q_s\mathbf{v}_s$ are:

$$\mathbf{F}_{st}^{T} = \overset{\text{indir}}{\mathbf{F}_{s}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau}} + \overset{\text{dir}}{\mathbf{F}_{st}^{\mathbf{E}_t}} + \overset{\text{dir}}{\mathbf{F}_{st}^{\mathbf{B}_t}}$$

Indirect force on $q_s\mathbf{v}_s$ $\quad$ $r \equiv r_{st}$ [Eqs. (10.38), (10.39), and (10.40)]

$$\begin{aligned}\mathbf{F}_{st}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau} &= \frac{d\mathbf{G}_{\ell st}^{\varepsilon_0\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau}}{dt} = \frac{1}{4\pi\varepsilon_0}\frac{1}{2c^2}\Bigg[\frac{\mathbf{a}_s}{r} - \frac{\mathbf{v}_s\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)}{r^3} \\ & + \frac{\left(\mathbf{a}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3} + \frac{\left(\mathbf{v}_s\cdot\mathbf{v}_t\right)\mathbf{r}_{st}}{r^3} - \frac{v_s^2\mathbf{r}_{st}}{r^3} + \frac{\mathbf{v}_t\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)}{r^3} \\ & - \frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^5} + \frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5}\Bigg]\frac{\left(q_tq_s+q_sq_t\right)}{2}\end{aligned} \quad (10.17)$$

Coulomb force of q_t on q_s :

$$\begin{aligned}\overset{\text{dir}}{\mathbf{F}_{st}^{\mathbf{E}_s}} &= \frac{1}{4\pi\varepsilon_0}\Bigg[-\frac{\mathbf{r}_{st}}{r^3} + \frac{1}{2c^2}\Bigg(-\frac{v_t^2\mathbf{r}_{st}}{r^3} + \frac{3\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5} \\ & + \frac{\mathbf{a}_t}{r} + \frac{\left(\mathbf{a}_t\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\Bigg)\Bigg]\left(q_tq_s+q_sq_t\right)/2\end{aligned} \quad (10.18)$$

Magnetic force of q_t on q_s :

$$\mathbf{F}_{st}^{\mathbf{B}_t} = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{2c^2}\left(-\frac{2\mathbf{v}_t\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)}{2r^3} + \frac{2\left(\mathbf{v}_s\cdot\mathbf{v}_t\right)\mathbf{r}_{st}}{2r^3}\right)\right]\left(q_tq_s+q_sq_t\right)/2 \quad (10.19)$$

Now write the above as the total force between two multivectors by adding Eqs. (10.17), (10.18), and (10.19). Total force is indicated by adding the superscript T to $\mathbf{F}$.

$$\begin{aligned}
&\mathbf{F}_{st}^{T} = \overset{\text{indir}}{\mathbf{F}_{st}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau}} + \overset{\text{dir}}{\mathbf{F}_{st}^{\mathbf{E}_t}} + \overset{\text{dir}}{\mathbf{F}_{st}^{\mathbf{B}_t}} = \\
&\frac{1}{4\pi\varepsilon_0}\left[-\frac{\mathbf{r}_{st}}{r^3} + \frac{1}{2c^2}\left(-\frac{\mathbf{v}_s(\mathbf{v}_t\cdot\mathbf{r}_{st})}{r^3} - \frac{\mathbf{v}_t(\mathbf{v}_s\cdot\mathbf{r}_{st})}{r^3} + \frac{3(\mathbf{v}_s\cdot\mathbf{v}_t)\mathbf{r}_{st}}{r^3}\right.\right. \\
&-\frac{v_t^2\mathbf{r}_{st}}{r^3} - \frac{v_s^2\mathbf{r}_{st}}{r^3} - \frac{3(\mathbf{v}_s\cdot\mathbf{r}_{st})(\mathbf{v}_t\cdot\mathbf{r}_{st})\mathbf{r}_{st}}{r^5} + \frac{3(\mathbf{v}_s\cdot\mathbf{r}_{st})^2\mathbf{r}_{st}}{r^5} \\
&\left.\left.+\frac{3(\mathbf{v}_t\cdot\mathbf{r}_{st})^2\mathbf{r}_{st}}{r^5} + \frac{\mathbf{a}_s}{r} + \frac{\mathbf{a}_s\cdot\mathbf{r}_{st}}{r^3} - \frac{\mathbf{a}_t}{r} - \frac{(\mathbf{a}_t\cdot\mathbf{r}_{st})\mathbf{r}_{st}}{r^3}\right)\right]\frac{(q_sq_t+q_tq_s)}{2} \qquad (10.20)
\end{aligned}$$

The above equations are obtained in Eqs. (10.4), (10.5), and (10.10) and replacing $\mathbf{r}_{st}$ by $-\mathbf{r}_{ts}$, and then we obtain the forces on $q_s\mathbf{v}_s$ associated with the joint motion $q_t\mathbf{v}_t$ and $q_s\mathbf{v}_s$.

Note that $\mathbf{F}_{ts}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau} + \mathbf{F}_{st}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau} \neq 0$. (Eqs. (10.15) and Eq. (10.17)). So the third law does not hold for the indirect forces alone. As previously mentioned, $\mathbf{F}_{ts}^{\mathbf{E}_s} + \mathbf{F}_{st}^{\mathbf{E}_t} \neq 0$ and also $\mathbf{F}_{ts}^{\mathbf{B}_s} + \mathbf{F}_{st}^{\mathbf{B}_t} \neq 0$. However, $\mathbf{F}_{ts}^{T} + \mathbf{F}_{st}^{T} = 0$ and the third law is satisfied.

Eq. (10.16) gives the force between any pair of bivectors of like kind. When q_s, q_t are electric charges, that is scalars, the forces are electromagnetic where $1/4\pi\varepsilon_0$ may be regarded as a proportionality constant in which case the quantities $\mathbf{E}_0$ and μ_0 are related by $\varepsilon_0\mu_0 = 1/c^2$. When q_s, q_t are space-time momentum vectors, that is gravitational charges, then the equation describes gravitational force between two masses. For arbitrary multivectors, re-label ε_0 and μ_0 with a superscript to indicate the kind of multivector. Thus for gravity, they are ε_0^g and μ_0^g where we still require $\varepsilon_0^g\mu_0^g = 1/c^2$. The 4π factor for ε_0 and μ_0 comes from normalizing the electric flux for $\mathbf{E}$ to 1. Flux associated with gravity is not normalized.

When q_s, q_t represent stationary gravitational charges, all velocity terms in Eq. (10.16) are zero, and then

$$\begin{aligned}
\mathbf{F}_{ts}^{T} &= \left(\frac{1}{4\pi\varepsilon_0^g}\frac{\mathbf{r}_{st}}{r^3}\right)(-m_sm_tc^2) \\
&= -\frac{c^2m_sm_t\mathbf{r}_{st}}{4\pi\varepsilon_0^g r^3} \qquad (10.21)
\end{aligned}$$

c^2 in Eq. (10.21) is the value of the gravitational charge at rest. Equate this to the formula for gravitational force

$$\begin{aligned} \frac{c^2 m_s m_t \mathbf{r}_{st}}{4\pi\varepsilon_0^g r^3} &= \frac{G m_s m_t \mathbf{r}_{st}}{r^3} \\ G &= \frac{c^2}{4\pi\varepsilon_0^g} \end{aligned} \tag{10.22}$$

Put the experimental value for G in Eq. (10.22) to obtain the value that ε_0^g must have in order for Eq. (10.22) to give the gravitational force between two masses $q_s = m_s$ and $q_t = m_t$. Also change ε_0 to ε_0^g.

$$\frac{1}{4\pi\varepsilon_0^g} = \frac{G}{c^2} = \frac{66.7 \times 10^{-10}}{9 \times 10^{10}} \qquad \frac{N\mathrm{meter}^2}{(\mathrm{kg})^2} \times \frac{\sec^2}{(\mathrm{meter})^2} = 7.4 \frac{N\sec^2}{(\mathrm{kg})^2}$$

For application of all electromagnetism formulas to gravity, replace q_s, q_t by space-time 4-momenta $\mathbf{P}_s$ and $\mathbf{P}_t$.

For electrical interaction $1/4\pi\varepsilon_0 = 8.987 \times 10^9 \ N \cdot m^2 C^{-2}$

To convert $\mathbf{F}_{ts}^T$ to gravitational interaction, put $q_s = P_s = m_s\left(\mathbf{e}_0 c + \mathbf{v}_s\right)$ and $q_t = P_t = m_t\left(\mathbf{e}_0 c + \mathbf{v}_t\right)$. We omit the relativistic factors $\gamma_s = 1/\left(1 - v_s^2/c^2\right)^{1/2}$ and $\gamma_t = 1/\left(1 - v_t^2/c^2\right)^{1/2}$ since we only require an accuracy of $1/c^2$. Also signs are changed to make all forces attractive.

$$P_t P_s = m_t m_s\left(-c^2 + \mathbf{v}_t \cdot \mathbf{v}_s\right) = -m_t m_s c^2\left(1 - \mathbf{v}_t \cdot \mathbf{v}_s / c^2\right)$$

Thus for gravity

$$\begin{aligned} \mathbf{F}_{ts}^T = \; & -m_t m_s G \left[\frac{\mathbf{r}_{st}}{r^3} + \frac{1}{2c^2} \left(\frac{\mathbf{v}_t\left(\mathbf{v}_s \cdot \mathbf{r}_{st}\right)}{r^3} + \underline{\frac{2\mathbf{v}_s\left(\mathbf{v}_t \cdot \mathbf{r}_{st}\right)}{r^3}} \right.\right. \\ & - \underline{\frac{3\left(\mathbf{v}_t \cdot \mathbf{v}_s\right)\mathbf{r}_{st}}{r^3}} + \underline{\frac{v_s^2 \mathbf{r}_{st}}{r^3}} + \frac{v_t^2 \mathbf{r}_{st}}{r^3} + \frac{3\left(\mathbf{v}_t \cdot \mathbf{r}_{st}\right)\left(\mathbf{v}_s \cdot \mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^5} \\ & - \frac{3\left(\mathbf{v}_t \cdot \mathbf{r}_{st}\right)^2 \mathbf{r}_{st}}{r^5} - \underline{\frac{3\left(\mathbf{v}_s \cdot \mathbf{r}_{st}\right)^2 \mathbf{r}_{st}}{r^5}} - \frac{\mathbf{a}_t}{r} \\ & \left.\left. - \frac{\left(\mathbf{a}_t \cdot \mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3} + \underline{\frac{\mathbf{a}_s}{r}} + \underline{\frac{\left(\mathbf{a}_s \cdot \mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}} \right) \right] \left(1 - \frac{\mathbf{v}_t \cdot \mathbf{v}_s}{c^2}\right) \end{aligned} \tag{10.23}$$

Also

$$
\begin{aligned}
\mathbf{F}_{st}^{T} &= -m_t m_s G\left[-\frac{\mathbf{r}_{st}}{r^3}+\frac{1}{2c^2}\left(-\frac{\mathbf{v}_s\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)}{r^3}-\frac{\mathbf{v}_t\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)}{r^3}\right.\right.\\
&\quad +\frac{3\left(\mathbf{v}_s\cdot\mathbf{v}_t\right)\mathbf{r}_{st}}{r^3}-\frac{v_t^2\mathbf{r}_{st}}{r^3}-\frac{v_s^2\mathbf{r}_{st}}{r^3}-\frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^5}\\
&\quad +\frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5}+\frac{3\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5}\\
&\quad \left.\left.+\frac{\mathbf{a}_s}{r}+\frac{\mathbf{a}_s\cdot\mathbf{r}_{st}}{r^3}-\frac{\mathbf{a}_t}{r}-\frac{\left(\mathbf{a}_t\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\right)\right]\left(1-\frac{\mathbf{v}_t\cdot\mathbf{v}_s}{c^2}\right)
\end{aligned}
\tag{10.24}
$$

Eq. (10.25), below, gives a group of the terms that contribute to the gravitational force of $q_s\mathbf{v}_s$ on $q_t\mathbf{v}_t$. Eq. (10.24) gives part of the terms that contribute to the gravitational force of $q_t\mathbf{v}_t$ on $q_s\mathbf{v}_s$. Note that in Eq. (10.25), collecting terms in v^2/c^2

$$
\begin{aligned}
-\frac{\mathbf{a}_t}{r} &\simeq \frac{v_t^2}{r^3}\mathbf{r}_{st} &\qquad -\frac{\left(\mathbf{a}_t\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3} &\simeq \frac{v_t^2\mathbf{r}_{st}}{r^3}\\
-\frac{\mathbf{a}_s}{r} &\simeq \frac{v_s^2}{r^3}\mathbf{r}_{st} &\qquad \frac{\left(\mathbf{a}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3} &\simeq \frac{v_s^2\mathbf{r}_{st}}{r^3}
\end{aligned}
$$

So that Eq. (10.25) may be written

$$
\begin{aligned}
\mathbf{F}_{ts} &= -m_t m_s G\frac{\mathbf{r}_{st}}{r^3}\left[1+\frac{1}{2c^2}\left(\frac{\mathbf{v}_t\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)}{r^3}+\underline{\frac{2\mathbf{v}_s\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)}{r^3}}-\underline{\frac{3\left(\mathbf{v}_t\cdot\mathbf{v}_s\right)\mathbf{r}_{st}}{r^3}}\right.\right.\\
&\quad +\underline{\frac{v_s^2\mathbf{r}_{st}}{r^3}}+\frac{v_t^2\mathbf{r}_{st}}{r^3}+\frac{3\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^5}-\frac{3\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5}\\
&\quad \left.\left.-\underline{\frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5}}+\frac{2v_t^2\mathbf{r}_{st}}{r^3}-\frac{2v_s^2\mathbf{r}_{st}}{r^3}\right)\right]\left(1-\frac{\mathbf{v}_t\cdot\mathbf{v}_s}{c^2}\right)
\end{aligned}
$$

Collecting the terms in v^2/c^2.

$$
\mathbf{F}_{ts}=-\frac{m_t m_s\mathbf{r}_{st}}{r^3}G\left(1+\frac{3}{2}\frac{v_t^2}{c^2}\right)
\tag{10.25}
$$

Thus G is increased to the effective value

$$
G\rightarrow G\left(1+\frac{3v_t^2}{2c^2}\right)
\tag{10.26}
$$

Torque density, Eq. (10.38), with $q_t=p_t$ and $q_s=p_s$, doubles this value so that the final effective value of G from these terms is $G\rightarrow G\left(1+\dfrac{3v_t^2}{c^2}\right)$.

10.6 Possible Alternate Grouping, Not Adopted

If we were to substitute $\mathbf{F}_{st}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau}$ for Eq. (10.15) rather than $\mathbf{F}_{st}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau}$, then

$$\mathbf{F}_{ts}^{T} = \mathbf{F}_{ts}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau} + \mathbf{F}_{ts}^{\mathbf{E}_s} + \mathbf{F}_{ts}^{\mathbf{B}_s}$$

The sum of the above is

$$\begin{aligned}\mathbf{F}_{ts}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau} &= \frac{d\mathbf{G}_{\ell ts}}{dt} = \frac{q_t q_s}{4\pi\varepsilon_0}\left[\frac{1}{2c^2}\left(\frac{\mathbf{a}_s}{r} - \frac{\mathbf{v}_s\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)}{r^3} + \frac{\mathbf{v}_t\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)}{r^3}\right.\right.\\ &+\frac{\left(\mathbf{a}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3} + \frac{\left(\mathbf{v}_s\cdot\mathbf{v}_t\right)\mathbf{r}_{st}}{r^3} - \frac{v_s^2\mathbf{r}_{st}}{r^3}\\ &+\frac{\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{v}_t}{r^3} - \frac{\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{v}_s}{r^3}\\ &\left.\left.-\frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^5} + \frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5}\right)\right]\end{aligned} \tag{10.27}$$

$$\begin{aligned}\mathbf{F}_{ts}^{\mathbf{E}_t} &= \frac{q_t q_s}{4\pi\varepsilon_0 c^2}\left[-\frac{\mathbf{r}_{st}}{r^3} - \frac{1}{2c^2}\left(\frac{v_t^2\mathbf{r}_{st}}{r^3} + \frac{3\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5}\right.\right. \quad \text{attractive Fig. (10.1c)}\\ &\left.\left.-\frac{\mathbf{a}_s}{r} - \frac{\left(\mathbf{a}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\right)\right]\end{aligned} \tag{10.28}$$

$$\mathbf{F}_{ts}^{\mathbf{B}_t} = \frac{q_t q_s}{4\pi\varepsilon_0}\left[\frac{1}{2c^2}\left(\frac{2\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{v}_t}{r^3} - \frac{2\left(\mathbf{v}_s\cdot\mathbf{v}_t\right)\mathbf{r}_{st}}{r^3}\right)\right] \quad \text{attractive Fig (10.1d)} \tag{10.29}$$

$$\begin{aligned}\mathbf{F}_{t}^{T} &= \mathbf{F}_{t}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau} + \mathbf{F}_{ts}^{\mathbf{E}_s} + \mathbf{F}_{ts}^{\mathbf{B}_s}\\ &= \frac{q_t q_s}{4\pi\varepsilon_0}\left\{-\frac{\mathbf{r}_{st}}{r^3} + \frac{1}{2c^2}\left[\frac{\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)\mathbf{v}_s}{r^3} - \frac{\left(\mathbf{v}_t\cdot\mathbf{v}_s\right)\mathbf{r}_{st}}{r^3}\right.\right.\\ &-\frac{v_s^2\mathbf{r}_{st}}{r^3} - \frac{v_t^2\mathbf{r}_{st}}{r^3} - \frac{\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{v}_s}{r^3} + \frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{2r^5}\\ &\left.\left.+\frac{3\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5} + \frac{\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{v}_t}{r^3} - \frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^5}\right]\right\}\end{aligned} \tag{10.30}$$

This substitution still satisfies the third law so $\mathbf{F}_{ts}^{T} + \mathbf{F}_{st}^{T} = 0$. However, the acceleration terms in $\mathbf{F}_{st}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau}$ cancel those contributed by $\mathbf{F}_{ts}^{\mathbf{E}_t}$. Thus the acceleration terms have no effect on $\mathbf{F}_{ts}^{T}$ or $\mathbf{F}_{t}^{T}$. That acceleration terms must certainly appear in the mutual interaction alone justifies the adoption of Eqs. (10.15) and (10.17) for the indirect force on $q_t\mathbf{v}_t$ and on $q_s\mathbf{v}_s$. We henceforth adopt Eq. (10.16) and Eq. (10.20) contain some of the correct interaction forces between $q_t\mathbf{v}_t$ and $q_s\mathbf{v}_s$.

One other combination is possible. Differentiate $\mathbf{E}_s \times \mathbf{B}_t + \mathbf{E}_t \times \mathbf{B}_s$ to obtain

$$\frac{d}{dt}\mathbf{E}_s \times \mathbf{B}_t + \mathbf{E}_s \times \frac{d}{dt}\mathbf{B}_t + \frac{d}{dt}\mathbf{E}_t \times \mathbf{B}_s + \mathbf{E}_t \times \frac{d}{dt}\mathbf{B}_s \tag{10.31}$$

From this group, we can make the tentative identification for the force on q_t

$$\mathbf{F}_{ts} = \int \mathbf{E}_s \times \frac{d\mathbf{B}_t}{dt} d\tau + \int \frac{d\mathbf{E}_t}{dt} \times \mathbf{B}_s d\tau \tag{10.32}$$

One may verify that the integrands in Eq. (10.33) are singular. Therefore this assignment is excluded.

Summary. The total force exerted on $q_t\mathbf{v}_t$ is

$$\begin{aligned}
&\mathbf{F}_{ts}^{T} = \mathbf{F}_{ts}^{\overset{\text{indir}}{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau}} + \mathbf{F}_{ts}^{\overset{\text{dir}}{\mathbf{E}_s}} + \mathbf{F}_{ts}^{\overset{\text{dir}}{\mathbf{B}_s}} \\
&\mathbf{F}_{ts}^{T} = \\
&\frac{q_t q_s}{4\pi\varepsilon_0 c^2}\left[\underline{\frac{\mathbf{r}_{ts}}{r^3}} + \frac{1}{2c^2}\left(\frac{\mathbf{v}_t\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)}{r^3} + \frac{\mathbf{v}_s\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)}{r^3} - \frac{3\left(\mathbf{v}_t\cdot\mathbf{v}_s\right)\mathbf{r}_{st}}{r^3}\right.\right. \\
&\underline{+\frac{v_s^2\mathbf{r}_{st}}{r^3}} + \frac{v_t^2\mathbf{r}_{st}}{r^3} + \frac{3\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^5} - \frac{3\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5} \\
&\left.\left.\underline{-\frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5}} + \frac{\mathbf{a}_t}{r} + \frac{\left(\mathbf{a}_t\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3} - \underline{\frac{\mathbf{a}_s}{r}} - \underline{\frac{\left(\mathbf{a}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}}\right)\right]\frac{\left(q_t q_s + q_s q_t\right)}{2}
\end{aligned} \tag{10.33}$$

Underlined terms are from $\mathbf{F}_{ts}^{\mathbf{E}_s}$ in Eq. (10.4).

The total force exerted on $q_s\mathbf{v}_s$ is

$$\begin{aligned}
&\mathbf{F}_{st}^{T} = \mathbf{F}_{st}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau} + \mathbf{F}_{st}^{\mathbf{E}_t} + \mathbf{F}_{st}^{\mathbf{B}_t} \\
&\mathbf{F}_{st}^{T} = \\
&\frac{q_s q_t}{4\pi\varepsilon_0}\left[-\underline{\frac{\mathbf{r}_{st}}{r^3}} - \frac{1}{2c^2}\left(\frac{\mathbf{v}_s\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)}{r^3} - \frac{\mathbf{v}_t\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)}{r^3} + \frac{3\left(\mathbf{v}_s\cdot\mathbf{v}_t\right)\mathbf{r}_{st}}{r^3}\right.\right. \\
&-\frac{v_s^2\mathbf{r}_{st}}{r^3} - \underline{\frac{v_t^2\mathbf{r}_{st}}{r^3}} - \frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)}{r^5} + \frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5} \\
&\left.\left.\underline{+\frac{3\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)^2\mathbf{r}_{st}}{r^5}} + \frac{\mathbf{a}_s}{r} + \frac{\left(\mathbf{a}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3} - \underline{\frac{\mathbf{a}_t}{r}} - \underline{\frac{\mathbf{a}_t\cdot\mathbf{r}_{st}}{r^3}}\right)\right]\frac{\left(q_t q_s + q_s q_t\right)}{2}
\end{aligned} \tag{10.34}$$

Thus $\mathbf{F}_{ts}^{T} + \mathbf{F}_{st}^{T} = 0$ and the third law is satisfied to terms in v^2/c^2. The terms that come from $\mathbf{E}_{st}$ are underlined.

The forces defined above come from the mutual terms, that is the last two terms in

$$\begin{aligned}\varepsilon_0 (\mathbf{E} \times \mathbf{B}) &= \varepsilon_0 (\mathbf{E}_s + \mathbf{E}_t) \times (\mathbf{B}_s + \mathbf{B}_t) \\ &= \varepsilon_0 [\mathbf{E}_s \times \mathbf{B}_s + \mathbf{E}_t \times \mathbf{B}_t + \mathbf{E}_s \times \mathbf{B}_t + \mathbf{E}_t \times \mathbf{B}_s]\end{aligned}$$

The first two terms $\varepsilon_0 (\mathbf{E}_s \times \mathbf{B}_s)$ and $\varepsilon_0 (\mathbf{E}_t \times \mathbf{B}_t)$ also deliver momentum and force into the field and thereby result in a force on $q_s\mathbf{v}_s$ and independently on $q_t\mathbf{v}_t$. These remain to be calculated.

10.7 Angular Momentum and Torques

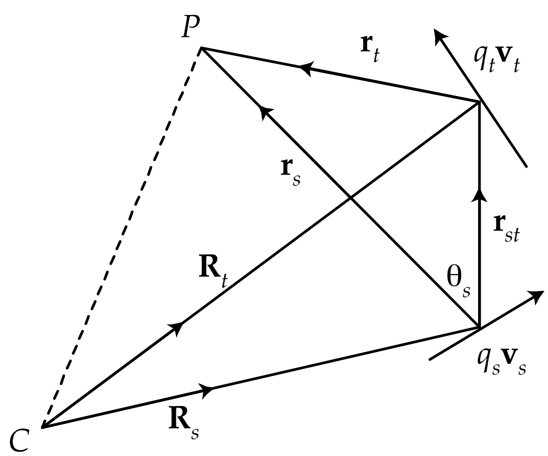

Fig. 10.2

The mutual angular momentum of $q_s\mathbf{v}_s$ and $q_t\mathbf{v}_t$ about point C, Fig. 10.2 is

$$\mathbf{G}_a = \frac{1}{4\pi\varepsilon_0}\frac{1}{2c^2}\left[\mathbf{R}_t \times \mathbf{G}_\ell^{\varepsilon_0 \int (\mathbf{E}_t \times \mathbf{B}_s) d\tau} + \mathbf{R}_s \times \mathbf{G}_\ell^{\varepsilon_0 \int (\mathbf{E}_s \times \mathbf{B}_t) d\tau}\right] \frac{(q_t q_s + q_s q_t)}{2} \quad (10.35)$$

That is,

$$\mathbf{G}_a = \frac{1}{4\pi\varepsilon_0}\frac{1}{2c^2}\left\{\mathbf{R}_t \times \left[\frac{\mathbf{v}_s}{r} + \frac{(\mathbf{v}_s \cdot \mathbf{r}_{st})\,\mathbf{r}_{st}}{r^3}\right] + \mathbf{R}_s \times \left[\frac{\mathbf{v}_t}{r} + \frac{(\mathbf{v}_t \cdot \mathbf{r}_{st})\,\mathbf{r}_{st}}{r^3}\right]\right\} \frac{(q_t q_s + q_s q_t)}{2} \quad (10.36)$$

From Eq. (10.37) we see that the portion of the indirect linear momentum, that is, the linear momentum contributed by the Poynting momentum vector, involving the velocity $\mathbf{v}_s$ of particle q_s is to be considered as located in the position of particle q_t which has the velocity $\mathbf{v}_t$ and vice versa (Page and Adams 1945).

The time derivative of $\mathbf{G}_a$ gives the total "indirect" torque about C.

$$\text{Torque} = \frac{d\mathbf{G}_a}{dt} \qquad \text{Indir Force on } q_s \qquad \text{Indir Force on } q_t$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{2c^2}\left[\mathbf{R}_t \times \frac{d}{dt}\mathbf{G}_\ell^{\varepsilon_0\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau} + \mathbf{R}_s \times \frac{d}{dt}\mathbf{G}_\ell^{\varepsilon_0\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau}\right]\frac{(q_tq_s + q_sq_t)}{2} \quad (10.37)$$

After integrating over all space, the torque about C contributed by the time rate of change of the mutual angular momentum about C is

$$\text{Torque} = \frac{d\mathbf{G}_a}{dt} =$$
$$\frac{1}{4\pi\varepsilon_0}\frac{1}{2c^2}\left\{\mathbf{R}_t \times \frac{d}{dt}\begin{bmatrix}\text{indir force on } q_s\\ \dfrac{\mathbf{v}_s}{r} + \dfrac{(\mathbf{v}_s\cdot\mathbf{r}_{st})\,\mathbf{r}_{st}}{r^3}\end{bmatrix} + \mathbf{R}_s \times \frac{d}{dt}\begin{bmatrix}\text{indir force on } q_t\\ \dfrac{\mathbf{v}_t}{r} + \dfrac{(\mathbf{v}_t\cdot\mathbf{r}_{st})\,\mathbf{r}_{st}}{r^3}\end{bmatrix}\right\}\frac{(q_tq_s + q_sq_t)}{2}$$
$$(10.38)$$

The term $\frac{d}{dt}\mathbf{G}^{\varepsilon_0\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau}$ was previously tentatively identified as the electromagnetic force on $q_t\mathbf{v}_t$. However, the above equation for the time rate of change of the angular momentum at $q_t\mathbf{v}_t$ indicates that $\mathbf{R}_s\times$ is the appropriate lever arm, not $\mathbf{R}_t\times$ as might intuitively be expected. If the identification Eq. (10.39) is correct then one is forced to conclude that, for the electromagnetic force, the appropriate lever arm is $\mathbf{R}_s$, the lever arm to $q_s\mathbf{v}_s$. This is perhaps counter intuitive but is a necessary assignment to retain acceleration terms in the overall force and torques. One could also argue that if torque is obtained by multiplying $q_t\mathbf{v}_t$ by $\mathbf{R}_t\times$ then the expression would be of the same form as the direct torques, each of which involve $\mathbf{R}_t\times$. This might be suspect since the field force involves a mixture of fields appearing in the Poynting vector, and one might not expect the indirect torque expressions to have the same form as the direct torque terms.

If we assign $\frac{d}{dt}\varepsilon_0\mathbf{G}_\ell^{\varepsilon_0\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau}$ as the electromagnetic force on $q_s\mathbf{v}_s$ rather than on $q_t\mathbf{v}_t$, then $\mathbf{R}_s$ would be the intuitively correct lever arm. In that case, however, no acceleration terms $\mathbf{a}_s$ and $\mathbf{a}_t$ appear in the electromagnetic forces on $q_s\mathbf{v}_s$ and $q_t\mathbf{v}_t$. This choice seems to be more counter intuitive than the assignment of the lever arm $\mathbf{R}_s$ in the second term of Eq. (10.37) and Eq. (10.38).

However, Eq. (10.37) says that the time rate of change of angular momentum is obtained by multiplying the force on $q_t\mathbf{v}_t$ by $\mathbf{R}_s$ which goes from C to the position of $q_s\mathbf{v}_s$, then add to it the direct force on $q_s\mathbf{v}_s$ multiplied by the lever arm going from C to the position of $q_t\mathbf{v}_t$.

Explicitly, referring back to Eq. (10.38), the term

$$\mathbf{R}_t \times \frac{d}{dt}\mathbf{G}_\ell^{\varepsilon_0 \int (\mathbf{E}_s \times \mathbf{B}_t) d\tau} = \mathbf{R}_t \times \text{force on } q_s\mathbf{v}_s$$

says that the torque associated with the force on $q_s\mathbf{v}_s$ is its value times $\mathbf{R}_t\times$, the lever arm to the other particle. Both particles jointly generate the force on $q_s\mathbf{v}_s$. Perhaps one must simply accept Eq. (10.39) in toto and not attempt to break it down. We did however break down the time rate of change of the total linear momentum by a rather convincing argument, to obtain the force on the individual sources.

There is no need to separate the total angular momentum and the total torque terms in all applications; however, in our treatment of forces the separation of the total linear momentum into two individual contributions is necessary in order to identify the force acting on each of the two particles, $q_t\mathbf{v}_t$ and $q_s\mathbf{v}_s$.

10.8 Evaluation of $F_{st}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau}$ and $F_{ts}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau}$

Following are details leading to the indirect force on $q_s\mathbf{v}_s$.

$$\begin{aligned}
\mathbf{G}^{\varepsilon_0\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau} &= \int \varepsilon_0\left(\mathbf{E}_t\times\mathbf{B}_s\right)d\tau \\
\mathbf{F}_{st}^{\varepsilon_0\frac{d}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau} &= \frac{d\mathbf{G}^{\varepsilon_0\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau}}{dt} \\
= & \frac{1}{4\pi\varepsilon_0}\frac{1}{2c^2}\frac{d}{dt}\left[\frac{\mathbf{v}_s}{r}+\frac{\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\right]\frac{\left(q_tq_s+q_sq_t\right)}{2} \\
= & \frac{1}{4\pi\varepsilon_0}\left\{\frac{1}{2c^2}\left[\frac{1}{r}\frac{d\mathbf{v}_s}{dt}+\mathbf{v}_s\frac{d}{dt}\left(\frac{1}{r}\right)\right.\right. \\
& +\left(\frac{d\mathbf{v}_s}{dt}\cdot\mathbf{r}_{st}\right)\frac{\mathbf{r}_{st}}{r^3}+\left(\mathbf{v}_s\cdot\frac{d\mathbf{r}_{st}}{dt}\right)\frac{\mathbf{r}_{st}}{r^3} \\
& \left.\left.+\frac{\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)}{r^3}\frac{d\mathbf{r}_{st}}{dt}+\frac{\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\frac{d}{dt}\left(\frac{1}{r^3}\right)\right]\right\}\frac{\left(q_tq_s+q_sq_t\right)}{2}
\end{aligned}\qquad(10.39)$$

The following derivatives are required to evaluate Eq. (10.39)

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{r_{st}}\right) &= \frac{d}{dt}\frac{1}{\left[\left(x_t-x_s\right)^2+\left(y_t-y_s\right)^2+\left(z_t-z_s\right)^2\right]^{1/2}} \\
&= -\frac{\left(x_t-x_s\right)\left(\dot{x}_t-\dot{x}_s\right)+\left(y_t-y_s\right)\left(\dot{y}_t-\dot{y}_s\right)+\left(z_t-z_s\right)\left(\dot{z}_t-\dot{z}_s\right)}{r^3} \\
&= -\frac{\mathbf{r}_{st}\cdot\left(\mathbf{v}_t-\mathbf{v}_s\right)}{r^3} = -\frac{\left(\mathbf{r}_{st}\cdot\mathbf{v}_t\right)}{r^3}+\frac{\left(\mathbf{r}_{st}\cdot\mathbf{v}_s\right)}{r^3}
\end{aligned}$$

$$\frac{d}{dt}\left(\frac{1}{r_{st}^3}\right)=\frac{d}{dt}\frac{1}{\left[(x_t-x_s)^2+(y_t-y_s)^2+(z_t-z_s)^2\right]^{3/2}}$$

$$=\quad -\frac{3}{2}\times\frac{2\left(x_t-x_s\right)\left(\dot{x}_t-\dot{x}_s\right)+2\left(y_t-y_s\right)\left(\dot{y}_t-\dot{y}_s\right)+2\left(z_t-z_s\right)\left(\dot{z}_t-\dot{z}_s\right)}{\left[\quad\right]^{5/2}}$$

$$\frac{d}{dt}\left(\frac{1}{r_{st}^3}\right)=-\frac{3\left(\mathbf{r}_{st}\cdot\mathbf{v}_t\right)}{r^5}+\frac{3\left(\mathbf{r}_{st}\cdot\mathbf{v}_s\right)}{r^5}$$

$$\frac{\left(\mathbf{v}_s\cdot\left(d\mathbf{r}_{st}/dt\right)\right)\mathbf{r}_{st}}{r^3}=\frac{\left(\mathbf{v}_s\cdot\mathbf{v}_t\right)\mathbf{r}_{st}}{r^3}-\frac{\left(\mathbf{v}_s\cdot\mathbf{v}_s\right)\mathbf{r}_{st}}{r^3}$$

$$\frac{\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)}{r^3}\frac{d\mathbf{r}_{st}}{dt}=\frac{\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{v}_t}{r^3}-\frac{\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{v}_s}{r^3}$$

$$\frac{\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\frac{d}{dt}\left(\frac{1}{r_{st}^3}\right)=-\frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}+\frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}$$

$$\frac{d\mathbf{r}_{st}}{dt}=\mathbf{v}_t-\mathbf{v}_s\qquad\frac{d^2\mathbf{r}_{st}}{dt^2}=\mathbf{a}_t-\mathbf{a}_s\qquad\frac{d\mathbf{v}_s}{dt}=\mathbf{a}_s\qquad\frac{d\mathbf{v}_t}{dt}=\mathbf{a}_t\tag{10.40}$$

We now substitute the preceding quantities into Eq. (10.39). The result is:

$$\mathbf{F}_{st}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau}=\frac{1}{4\pi\varepsilon_0}\left\{\frac{1}{2c^2}\left[\frac{\mathbf{a}_s}{r}+\mathbf{v}_s\left[\frac{\left(\mathbf{r}_{st}\cdot\mathbf{v}_s\right)}{r^3}+\frac{-\left(\mathbf{r}_{st}\cdot\mathbf{v}_t\right)}{r^3}\right]\right.\right.$$
$$+\left(\mathbf{a}_s\cdot\mathbf{r}_{st}\right)\frac{\mathbf{r}_{st}}{r^3}+\mathbf{v}_s\left(\mathbf{v}_t-\mathbf{v}_s\right)\frac{\mathbf{r}_{st}}{r^3}+\frac{\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\left(\mathbf{v}_t-\mathbf{v}_s\right)}{r^3}$$
$$+\frac{\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\left[-\frac{3\left(\mathbf{r}_{st}\cdot\mathbf{v}_t\right)}{r^3}+\frac{3\left(\mathbf{r}_{st}\cdot\mathbf{v}_s\right)}{r^3}\right]\frac{\left(q_tq_s+q_sq_t\right)}{2}$$

That is

$$\mathbf{F}_{st}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_t\times\mathbf{B}_s)d\tau}=\frac{d\mathbf{G}^{\varepsilon_0(\mathbf{E}_t\times\mathbf{B}_s)}}{dt}=\frac{1}{4\pi\varepsilon_0c^2}\frac{1}{2}\left[\frac{\mathbf{a}_s}{r}-\frac{\mathbf{v}_s\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)}{r^3}+\frac{\mathbf{v}_s\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)}{r^3}\right.$$
$$+\frac{\left(\mathbf{a}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}+\frac{\left(\mathbf{v}_s\cdot\mathbf{v}_t\right)\mathbf{r}_{st}}{r^3}-\frac{\left(\mathbf{v}_s\cdot\mathbf{v}_s\right)\mathbf{r}_{st}}{r^3}$$
$$+\frac{\mathbf{v}_t\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)}{r^3}-\frac{\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{v}_s}{r^3}$$
$$\left.-\frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)}{r^3}+\frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\right]\frac{\left(q_tq_s+q_sq_t\right)}{2}\tag{10.41}$$

Likewise

$$\mathbf{F}_{ts}^{\varepsilon_0 \frac{d}{dt}\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau} = \frac{d\mathbf{G}^{\varepsilon_0\int(\mathbf{E}_s\times\mathbf{B}_t)d\tau}}{dt} = \frac{1}{4\pi\varepsilon_0}\left\{\frac{1}{2c^2}\left[\frac{\mathbf{a}_t}{r} + \frac{\mathbf{v}_t\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)}{r^3} - \frac{\mathbf{v}_t\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)}{r^3}\right.\right.$$
$$+\frac{\left(\mathbf{a}_t\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3} - \frac{\left(\mathbf{v}_s\cdot\mathbf{v}_t\right)\mathbf{r}_{st}}{r^3} + \frac{\left(\mathbf{v}_s\cdot\mathbf{v}_s\right)\mathbf{r}_{st}}{r^3}$$
$$-\frac{\mathbf{v}_s\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)}{r^3} + \frac{\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)\mathbf{v}_t}{r^3}$$
$$\left.-\frac{3\left(\mathbf{v}_t\cdot\mathbf{r}_{st}\right)\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)}{r^3} + \frac{3\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\left(\mathbf{v}_s\cdot\mathbf{r}_{st}\right)\mathbf{r}_{st}}{r^3}\right]\frac{\left(q_tq_s + q_sq_t\right)}{2} \tag{10.42}$$