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GIBBS ALGEBRA AND CLIFFORD ALGEBRA

Our initial purpose is to show that space-time algebra suggests the possibility of a field associated with each of the basic entities in space-time algebra, namely a scalar, a vector, a bivector, a trivector and a quadvector. The mediator(s)of the field in each case may be massless, (photon-like), as in the Coulomb $1/r$ potential or massive as in the Yukawa potential, $Y = qe^{-M/r}/r$, thereby doubling the number of possible candidates.

It is our hypothesis that the multivectors of space-time algebra describe properties of space-time and tell us that all phenomena in nature can be explained by the algebra and all phenomena predicted by the algebra probably exist in nature. Since we will derive the algebra from the Pythagorean Theorem, one might say that all the properties of nature follow from the Pythagorean Theorem. Multivectors display their properties when at rest and when in motion; similar to electric charge which is itself effectively a scalar multivector. The transport or carrier algebra must be different from the multivector or property being transported. Therefore two commuting algebras are required in order to distinguish between the carrier algebra, that is, the velocity algebra and the property algebra[1] characterized by the various multivectors.

The special case of Clifford algebra consisting of three space-like unit vectors and one time-like unit vector has been called space time algebra by Hestenes (1971, 1982,

[1] These terms were first introduced by Greider and his student Alfred Differ (Differ 1991).

plus other publications) and as stated appears to be sufficient to describe nature, that is, the 3 space and one time dimensional world of which we are a part.

One may phrase our basic thesis with a question: Does physics follow from the Pythagorean theorem? If we ask for the fundamental property of space the answer would be that distances add quadratically. For example, referring to Fig. 1.1a or Fig. 1.1b, $a^2 + b^2 = c^2$ by the Pythagorean theorem (Pythagoras about 600 B.C.). The importance of the Pythagorean Theorem was first emphasized by Fermat's Last Theorem[2] which states that the equation $a^n + b^n = c^n$ is satisfied for integer values of a, b, c only when $n = 2$. If it were satisfied otherwise then our contention that it represents a unique property of space would not hold. Two examples for which the relationship is true are: $a,\ b,\ c\ =\ 3,\ 4,\ 5; 5, 12, 13$. For a description of the many properties of the Pythagorean triangle, see Sierpinski's *Pythagorean Triangles* (2003).

The implications of this and its connection with other aspects of nature was not appreciated until the late 19th century and even into the 20th century when the associated Clifford algebra and some of its consequences were developed by various authors, particularly Hestenes and later Greider (1984).

In the 19th century physicists and mathematicians sought a formalism or algebra for rotating and otherwise manipulating directed line segments. One approach culminated in the work of Hamilton and his quaternions. Clifford introduced his Clifford algebra by forming the direct product of two quaternion algebras. Willard Gibbs developed "vector analysis" a few years after Clifford's work. Clifford died at the early age of 33 in 1879, a few years after defining his algebra, so there was no protagonist to promote Clifford algebra. Gibbs' treatment of vectors, called vector analysis, was adopted by the scientific community as a compact and efficient way of manipulating directed line segments, that is, vectors. Gibbs defined a scalar product and a cross product. The Gibbs definition of the cross product does not extend to higher dimensions. The analogous wedge product in Clifford algebra, however, does so. Also Clifford algebra describes the space-time concepts that arose in Einstein's work on special relativity, which does not fit in the Gibbs formalism.

In this paper we describe gravity by using two commuting space-time algebras. The idea of describing gravity by introducing two commuting space-time algebras was first suggested and pursued by Kenneth Greider and his student Fred Morris (Morris 1983). However, many essential terms were not included in the analysis. That curved space is an essential part of the input also was overlooked. A complete and

[2] Fermat's Last Theorem (1601 to 1665): The equation $z^n = x^n + y^n$ has no integer solutions for $n > 2$. Fermat claimed he had a simple proof, but it was never recorded. The theorem was finally proven by Andre Wiles in 1994 utilizing a long, complicated procedure.

more careful treatment, as will be seen, yields the same results as general relativity for gravitational phenomena. In addition, a simple connection is provided between gravity and the rest of physics; for example, the gravitational field can be simply quantized.

We now show how the Pythagorean theorem gives rise to a two-dimensional algebra. The algebra may be extended to higher dimensions. Since the Pythagorean theorem reflects a fundamental property of space, we argue that the entities generated by the algebra may describe properties of space. We begin by obtaining a two-dimensional Gibbs algebra and illustrate how it differs from Clifford algebra.

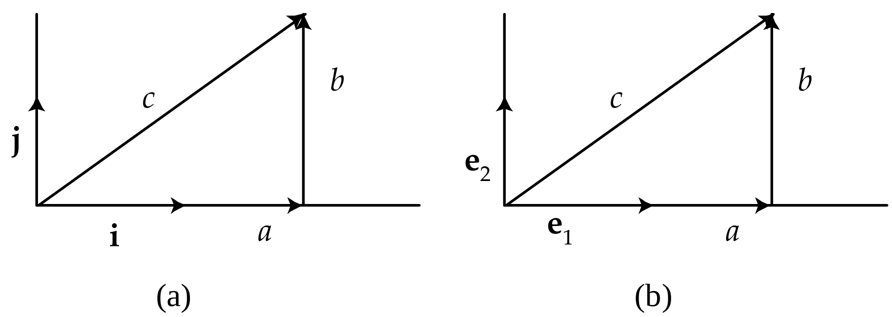

Fig. 1.1. Two dimensional example illustrating the difference between Clifford Algebra and Gibbs Algebra.

In Gibbs' notation, $\mathbf{i}$, $\mathbf{j}$ are unit vectors that define a pair of orthogonal axes as shown in Fig. 1.1a. Then form the product:

$$\mathbf{c} \cdot \mathbf{c} = c^2 = (a\mathbf{i} + b\mathbf{j}) \cdot (a\mathbf{i} + b\mathbf{j}) = a^2\mathbf{i} \cdot \mathbf{i} + ab\,(\mathbf{i} \cdot \mathbf{j} + \mathbf{j} \cdot \mathbf{i}) + b^2\mathbf{j} \cdot \mathbf{j}$$

Gibbs defined $\mathbf{c} \cdot \mathbf{c} = c^2$; $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$; $\mathbf{i} \cdot \mathbf{j} = 0$; $\mathbf{j} \cdot \mathbf{i} = 0$ to obtain $c^2 = a^2 + b^2$.

To go further, Gibbs also introduced the cross product symbol $\times$ so that $\mathbf{i} \times \mathbf{j}$ is a unit vector perpendicular to the plane of $\mathbf{i}$ and $\mathbf{j}$ and has the direction of a right hand screw turning from $\mathbf{i}$ to $\mathbf{j}$. This defines a third axis, $\mathbf{k}$ perpendicular to $\mathbf{i}$ and $\mathbf{j}$. More generally, for two arbitrary vectors $\mathbf{a} \times \mathbf{b} = \mathbf{n}ab\sin\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$; $\mathbf{n}$ is a unit vector in the direction stated. The dot and cross products are the two inputs that lead to the study of vector analysis.

To introduce Clifford algebra, let $\mathbf{c} = a\mathbf{e}_1 + b\mathbf{e}_2$ where, to change notation, the unit vectors $\mathbf{i}$, $\mathbf{j}$ have been replaced by $\mathbf{e}_1$ and $\mathbf{e}_2$. Then

$$\begin{aligned}
\mathbf{c}^2 = c^2 &= (a\mathbf{e}_1 + b\mathbf{e}_2)\,(a\mathbf{e}_1 + b\mathbf{e}_2) \\
c^2 &= a^2\mathbf{e}_1\mathbf{e}_1 + ab\mathbf{e}_1\mathbf{e}_2 + ba\mathbf{e}_2\mathbf{e}_1 + b^2\mathbf{e}_2\mathbf{e}_2 \\
c^2 &= a^2\mathbf{e}_1\mathbf{e}_1 + ab\,(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) + b^2\mathbf{e}_2\mathbf{e}_2
\end{aligned}$$

If the Pythagorean Theorem is to hold, let the unit vectors have the property

$$\mathbf{e}_1\mathbf{e}_1 = \mathbf{e}_2\mathbf{e}_2 = 1 \text{ and } \quad \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = 0$$

that is, $\mathbf{e}_1$, $\mathbf{e}_2$ anticommute. Then $c^2 = a^2 + b^2$. Thus the Pythagorean Theorem in this formalism yields a two-dimensional Clifford algebra formed by two *generators* $\mathbf{e}_1$, $\mathbf{e}_2$ which form an algebra. The elements of the algebra consist of the generators plus all possible independent entities obtained by forming the direct products of the generators. Thus, the 2-dimensional algebra consists of the entities:

1 ;	$\mathbf{e}_1,\ \mathbf{e}_2$;	$\mathbf{e}_1\mathbf{e}_2$.
scalar	vectors	bivector

A convenient procedure for generating the multivectors is to evaluate:

$$(1 + \mathbf{e}_1)(1 + \mathbf{e}_2) = 1 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_1\mathbf{e}_2$$

The entities are called multivectors or, specifically in this case, a scalar, two vectors, and a bivector.

Note that

$$(\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_1\mathbf{e}_2) = -(\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_2\mathbf{e}_1) = -1$$

The products $\mathbf{e}_1\mathbf{e}_2$ etc. are called direct or associative products. Three space dimensions plus time are described in the text.

The direct product, is all that is required in Clifford algebra, but it proves convenient to recognize that the direct product of two multivectors may be expressed by the sum of two parts given by

$$\mathbf{ab} = (\mathbf{ab} + \mathbf{ba})/2 + (\mathbf{ab} - \mathbf{ba})/2 = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$

The first term is called the scalar, inner, or dot product and the second the wedge or outer product and written as shown. The two products are useful to recognize since they have distinct and different properties.

Common to all interactions are the Coulomb force or the Yukawa force. The form of the Coulomb inverse r^2 force law, or $1/r$ potential, may be deduced from the idea that the mediators of particle interactions (flux) that follow geodesics[3] and are to be identified with the $1/r^2$ attenuation of geodesic density originating at a geodesic source. The Yukawa potential for massive particles is obtained by placing an attenuation factor in the Coulomb potential.

[3] A geodesic is the shortest distance between two interacting particles which we later arbitrarily call a source particle and a test particle, roles interchangeable.

In order to deal with rates of change, we introduce the four-dimensional gradient operator (an ordering parameter):

$$\square = \mathbf{e}_0 \frac{\partial}{c\partial t} + \mathbf{e}_1 \frac{\partial}{\partial x} + \mathbf{e}_2 \frac{\partial}{\partial y} + \mathbf{e}_3 \frac{\partial}{\partial z}$$

When describing forces between multivectors, one must recognize that they can appear in one way when the multivectors are at rest and in a different way when they are in motion. This requires the introduction of an algebra that describes velocity and acceleration and a separate algebra that describes the quantity being transported. The two algebras will commute.

It is our job to find the basic properties of space and describe what can exist in that space.

The squares of the vectors in Clifford Algebra, in general, do not have to be +1. They can be -1. AClifford Algebra velocity multivector is given by

$$\mathbf{V} = \mathbf{e}_0 c + \mathbf{e}_1 v_x + \mathbf{e}_2 v_y + \mathbf{e}_3 v_z$$

where

$$\begin{aligned} \mathbf{e}_0\mathbf{e}_1 &= -\mathbf{e}_1\mathbf{e}_0, \quad \mathbf{e}_0\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_0, \quad \mathbf{e}_0\mathbf{e}_3 = -\mathbf{e}_3\mathbf{e}_0 \\ \mathbf{e}_0^2 &= -1, \quad \mathbf{e}_1^2 = 1, \quad \mathbf{e}_2^2 = 1, \quad \mathbf{e}_3^2 = 1 \end{aligned}$$